Controlling Singular Values with Semidefinite Programming

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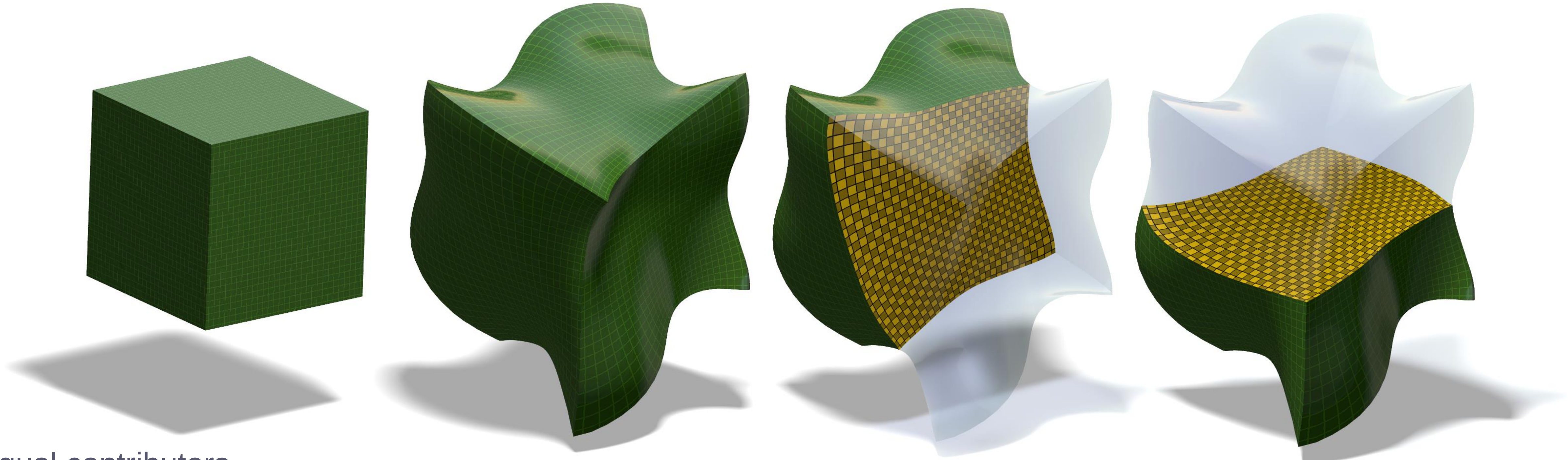


* equal contributors

Singular Values

A

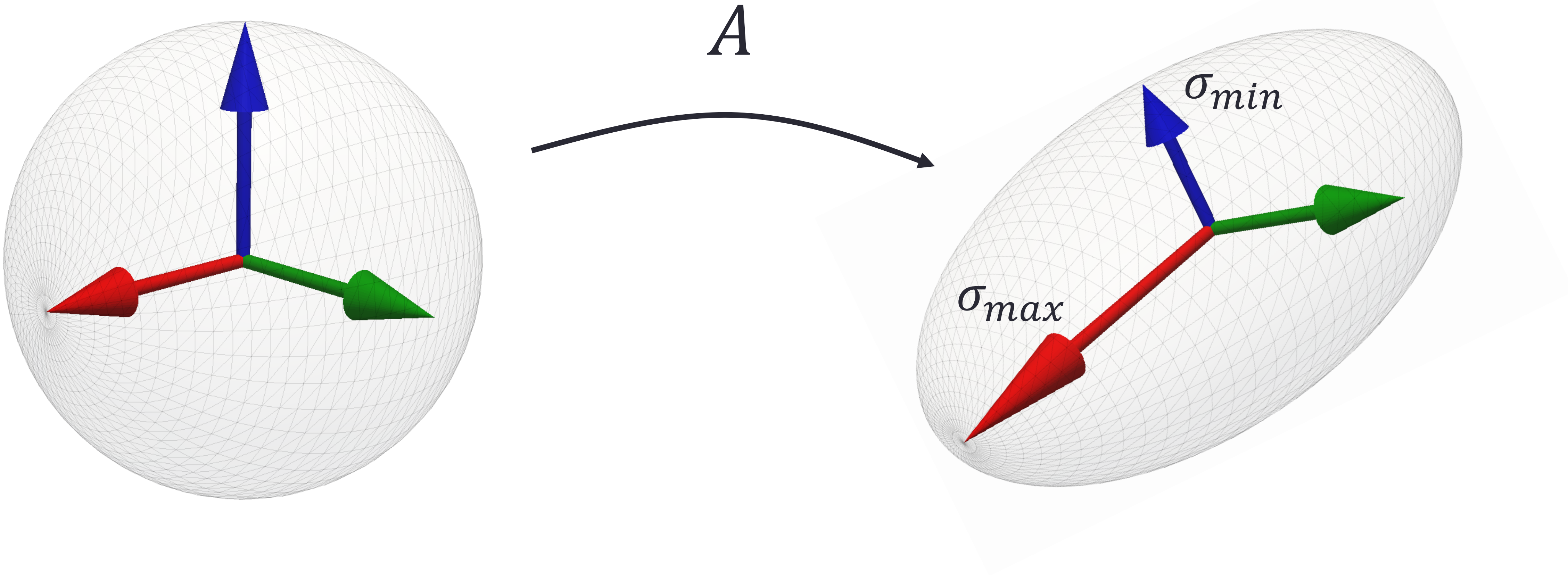
σ_2 σ_4

σ_1

σ_3

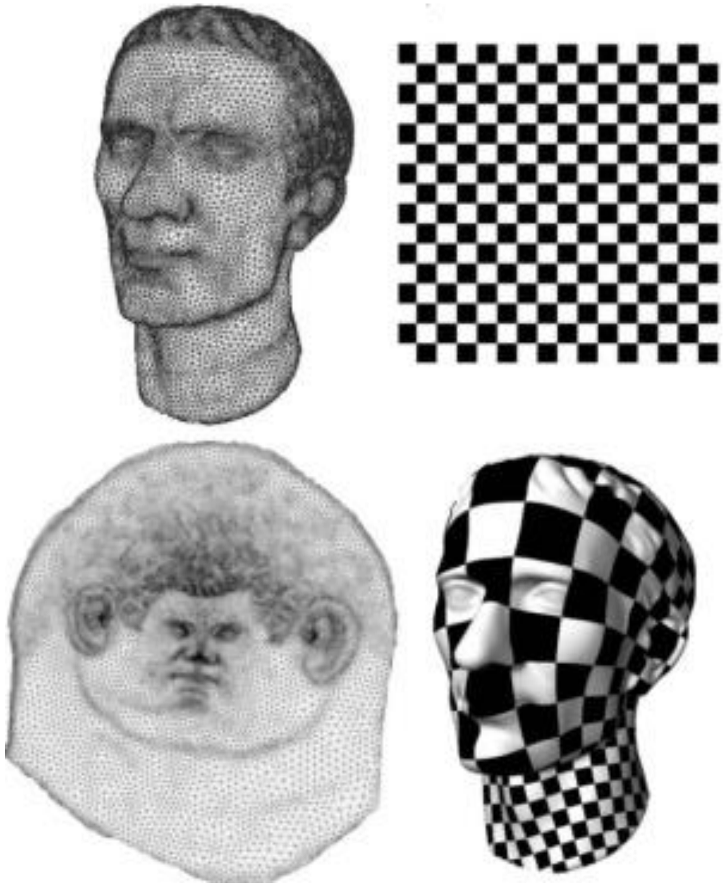
σ_5

Singular Values



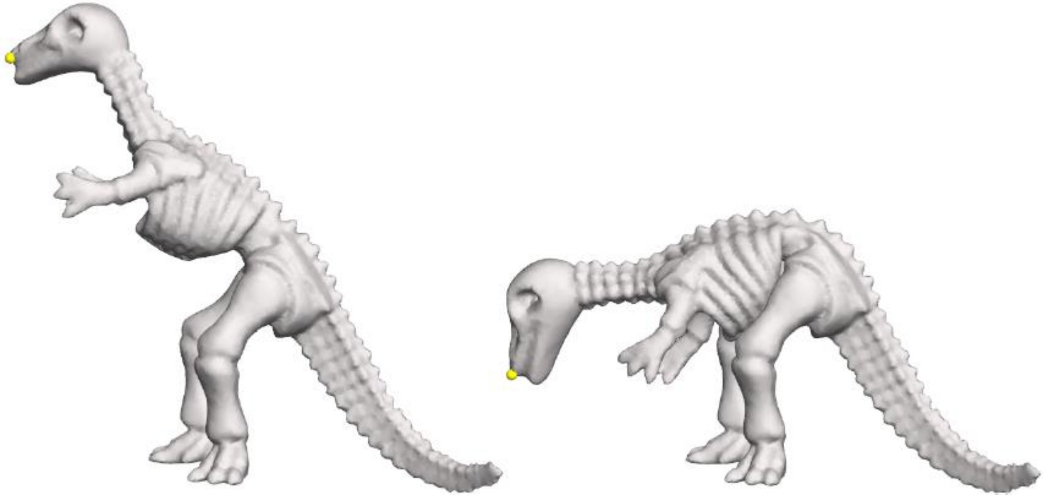
Motivation

Least Squares Conformal Mappings



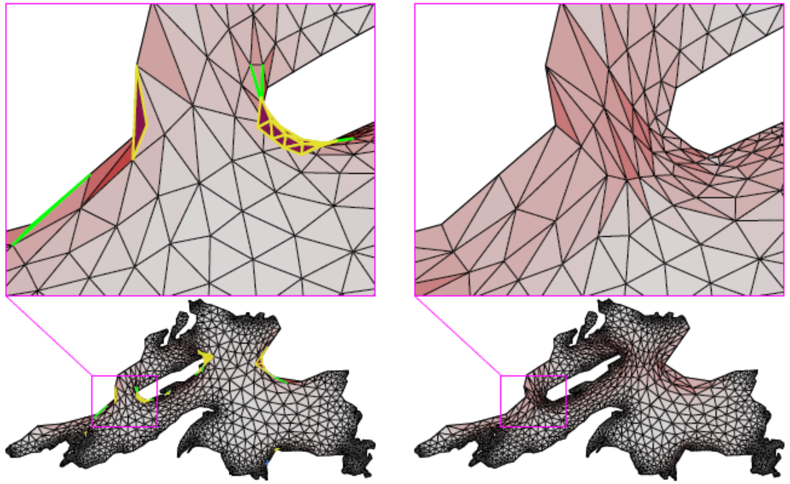
[Lévy et al. 2002]

As-Rigid-As-Possible



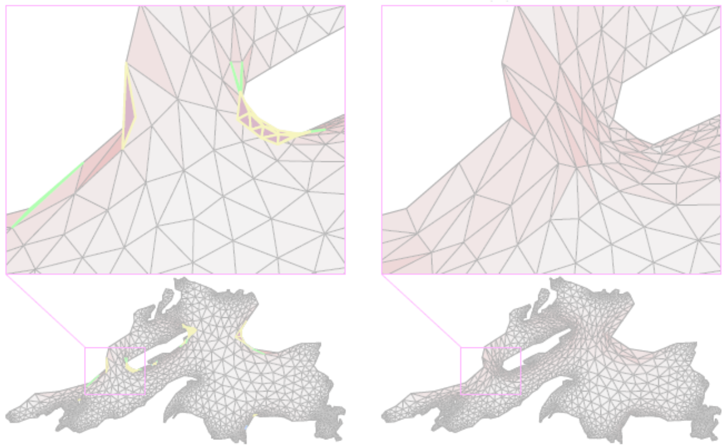
[Sorkine and Alexa 2007]

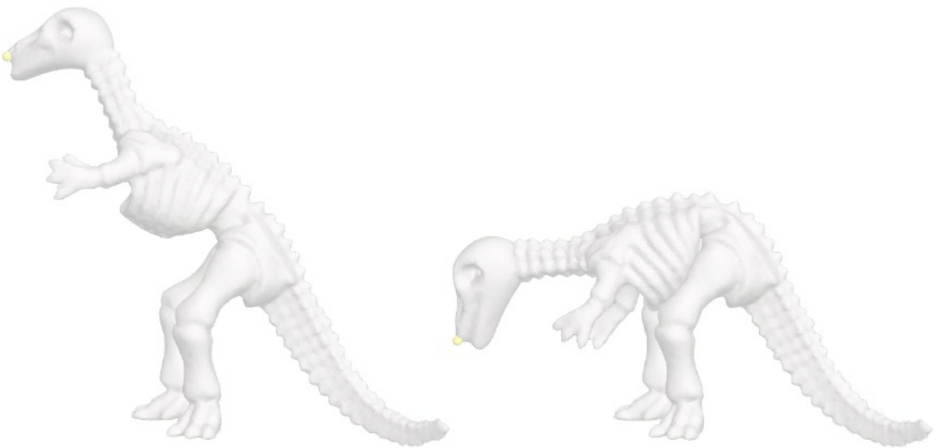
Bounded Distortion Mappings

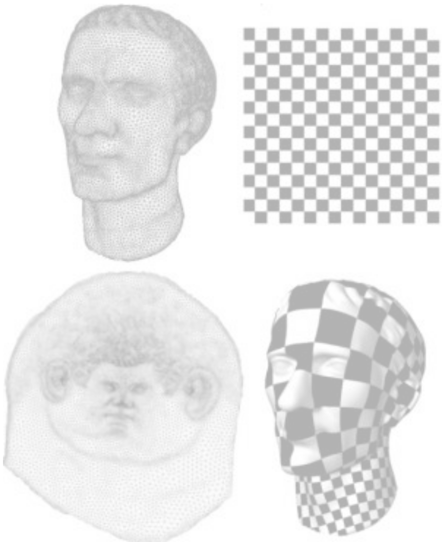


[Lipman 2012]

Common Perspective







This Work

- Optimization of linear transformations = **Matrices**
- Explicitly involving **singular values**
  - Constraints
  - Energies

Constrained Matrix Optimization

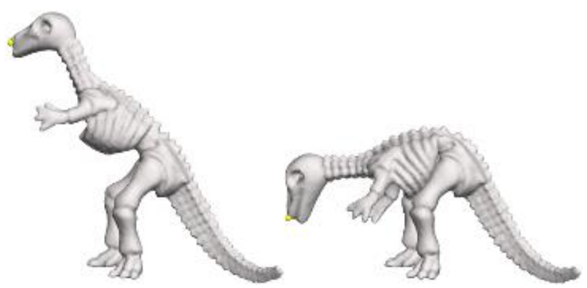
$$\min_{A \in \mathbb{R}^{n\times n}} \; f(A)$$

$$s.t. \; g_i(A) \leq 0$$

$A = (A_1, \dots, A_m)$

Energy

$$\|A - R_A\|_F$$

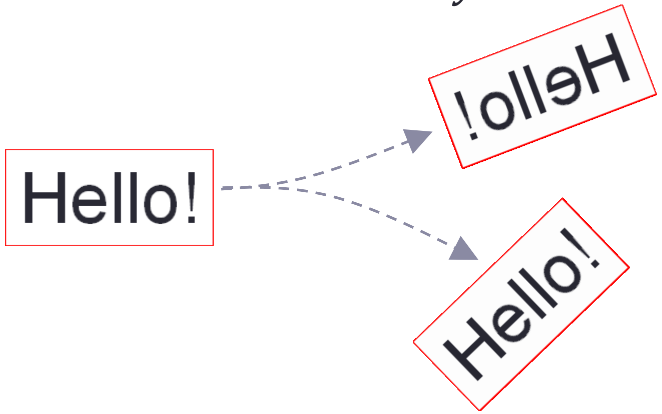


Constraints

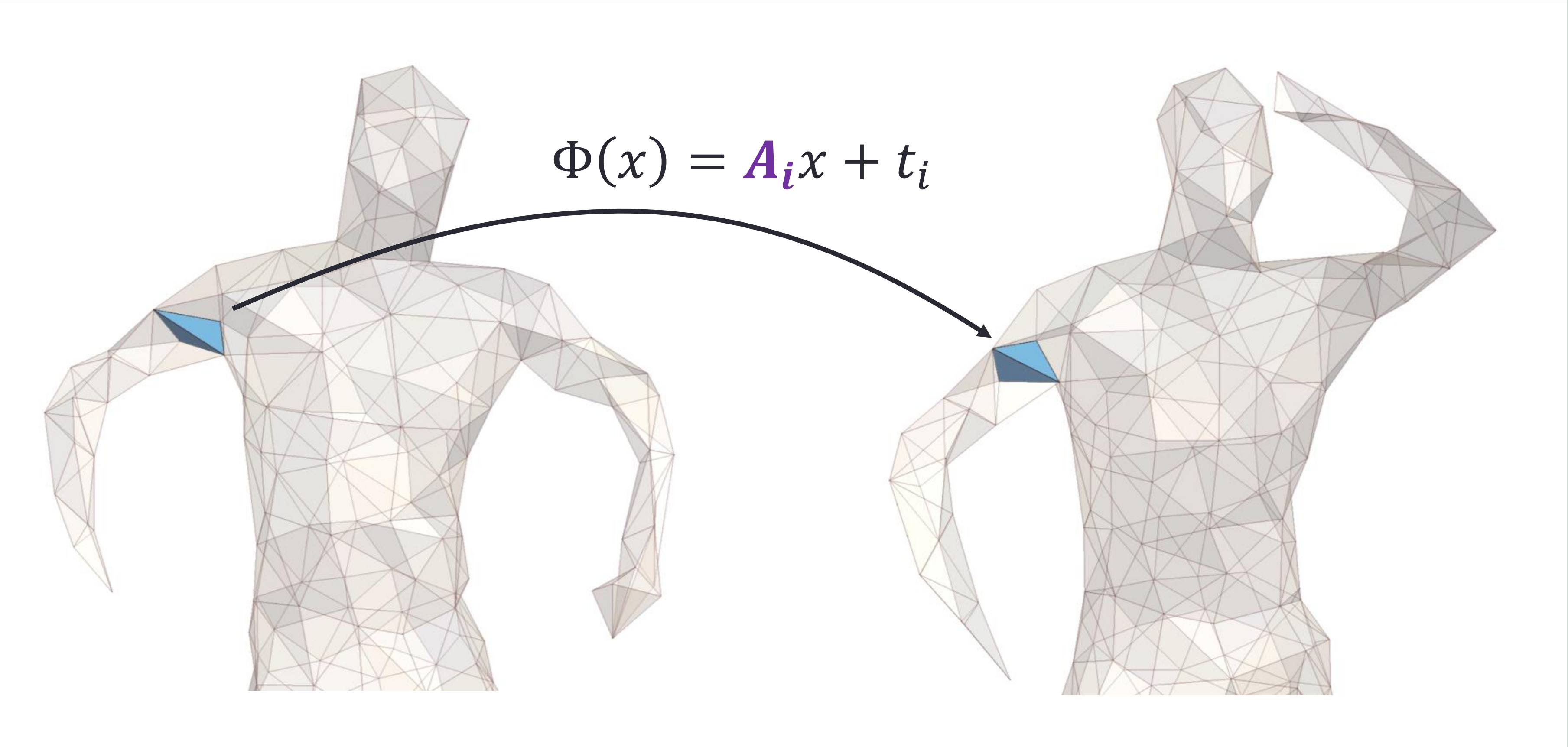
$$Ax = y$$

$$A^T A = I$$

$$\det(A) > 0$$



Constrained Matrix Optimization



Optimization of Singular Values

$$\min_{A \in \mathbb{R}^{n \times n}} \; f(A) \quad \text{Energy}$$

$$s.t. \; g_i(A) \leq 0 \quad \text{Constraints}$$

- Directly optimize singular values $\sigma_1, \ldots, \sigma_n$

This Work

$$\min_{A \in \mathbb{R}^{n \times n}} \quad f(A, \sigma_{\min}(A), \sigma_{\max}(A))$$
$$s.t. \quad g_i(A, \sigma_{\min}(A), \sigma_{\max}(A)) \leq 0$$

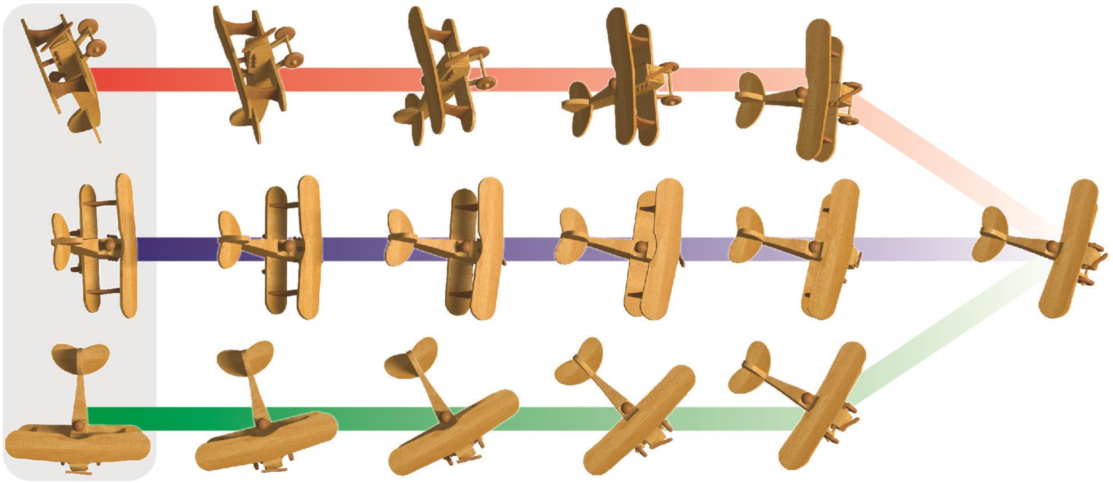
Optimization problems explicitly expressed in terms of **extremal SVs**

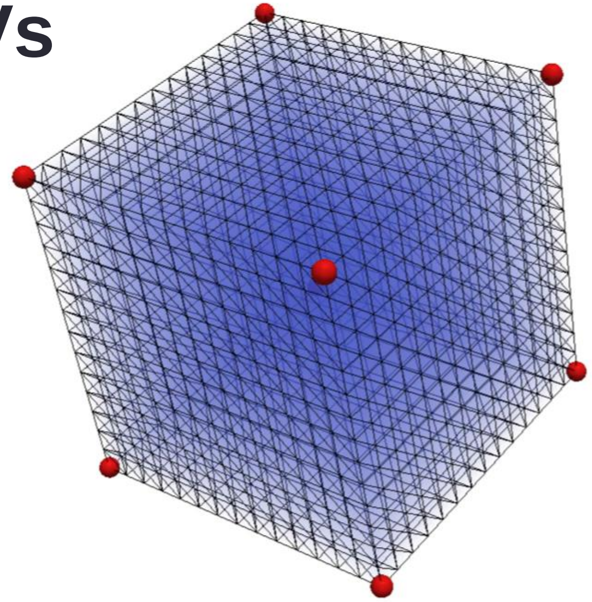
- Challenging
  - Non-linear
  - Non-convex
  - No closed form

This Work

$$\min_{A \in \mathbb{R}^{n \times n}} \quad f(A, \sigma_{\min}(A), \sigma_{\max}(A))$$
$$s.t. \quad g_i(A, \sigma_{\min}(A), \sigma_{\max}(A)) \leq 0$$

Optimization problems explicitly expressed in terms of **extremal SVs**





Key Results

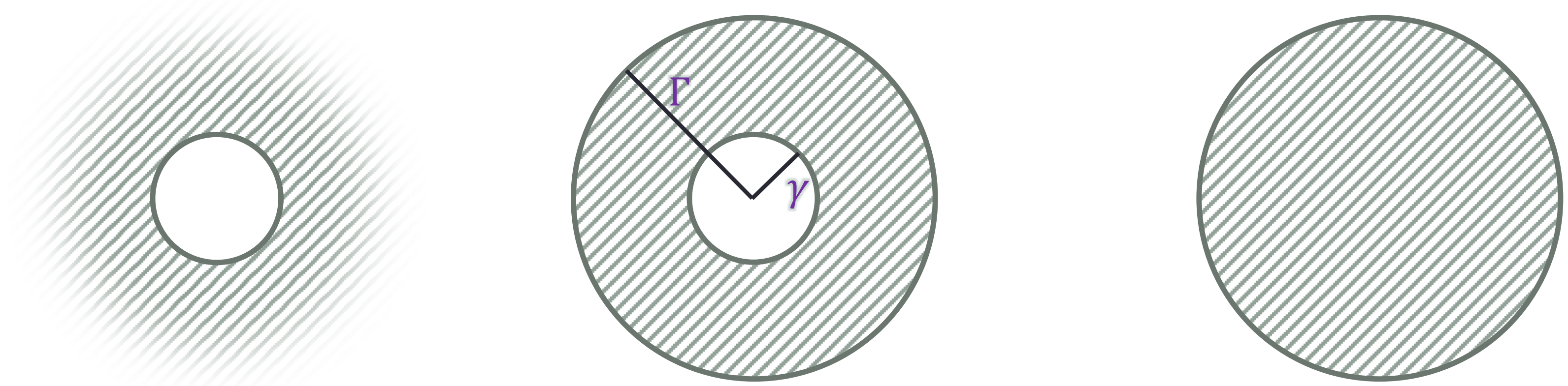
$$\min_{A \in \mathbb{R}^{n \times n}} \quad f(A, \sigma_{\min}(A), \sigma_{\max}(A))$$
$$s.t. \quad g_i(A, \sigma_{\min}(A), \sigma_{\max}(A)) \leq 0$$

- Can be reduced to

$$\gamma \leq \sigma_{\min}(A) \leq \sigma_{\max}(A) \leq \Gamma$$

Non-convex

Convex



Key Results

$$\begin{aligned} \min_{A \in \mathbb{R}^{n \times n}} \quad & f(A, \sigma_{\min}(A), \sigma_{\max}(A)) \\ s.t. \quad & g_i(A, \sigma_{\min}(A), \sigma_{\max}(A)) \leq 0 \end{aligned}$$

- Can be reduced to

$$\gamma \leq \sigma_{\min}(A) \leq \sigma_{\max}(A) \leq \Gamma$$

$$\Downarrow$$

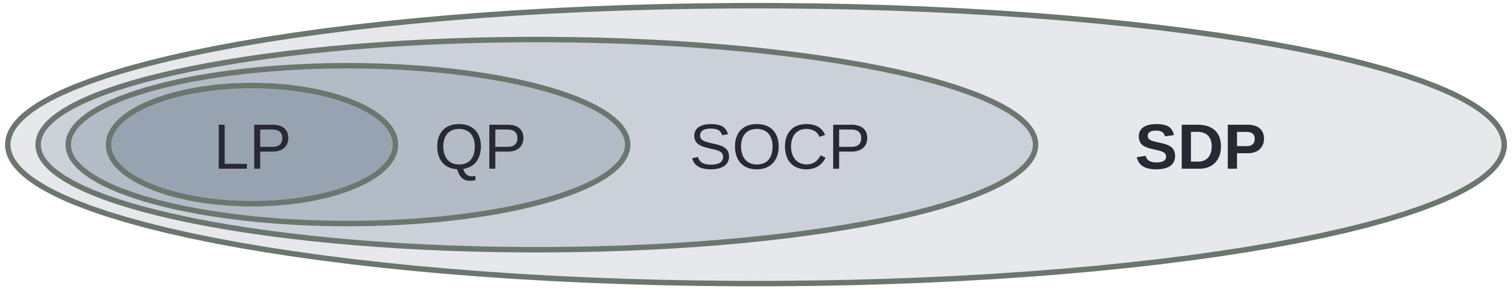
Semidefinite Programming

Optimal in the convex regime

Key Results

Semidefinite Programming (**SDP**)

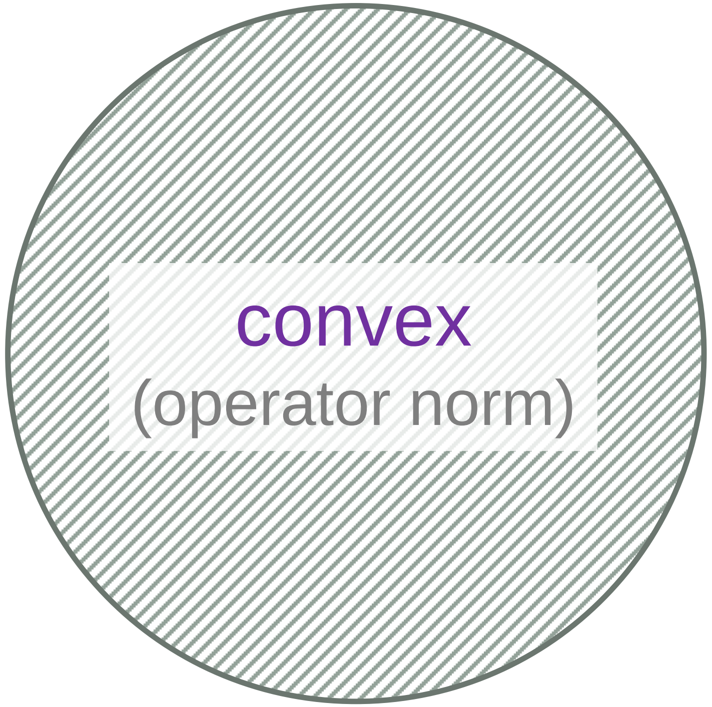
- SDP $\Leftrightarrow$ optimization with $\succcurlyeq$ constraints
- $A \succcurlyeq 0$ $\Leftrightarrow$ A is positive semidefinite



The natural class for dealing with singular values

Bounding SVs from above

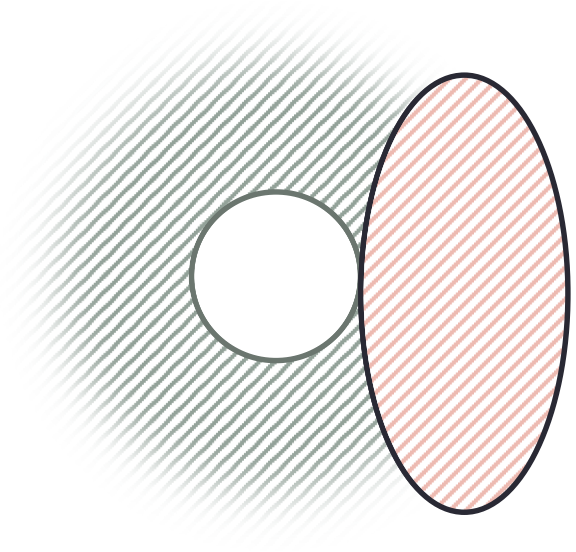
$$\sigma_{\max}(A) \leq \Gamma \quad \Leftrightarrow \quad \begin{pmatrix} \Gamma I & A \\ A^T & \Gamma I \end{pmatrix} \succcurlyeq 0$$



Bounding SVs from below

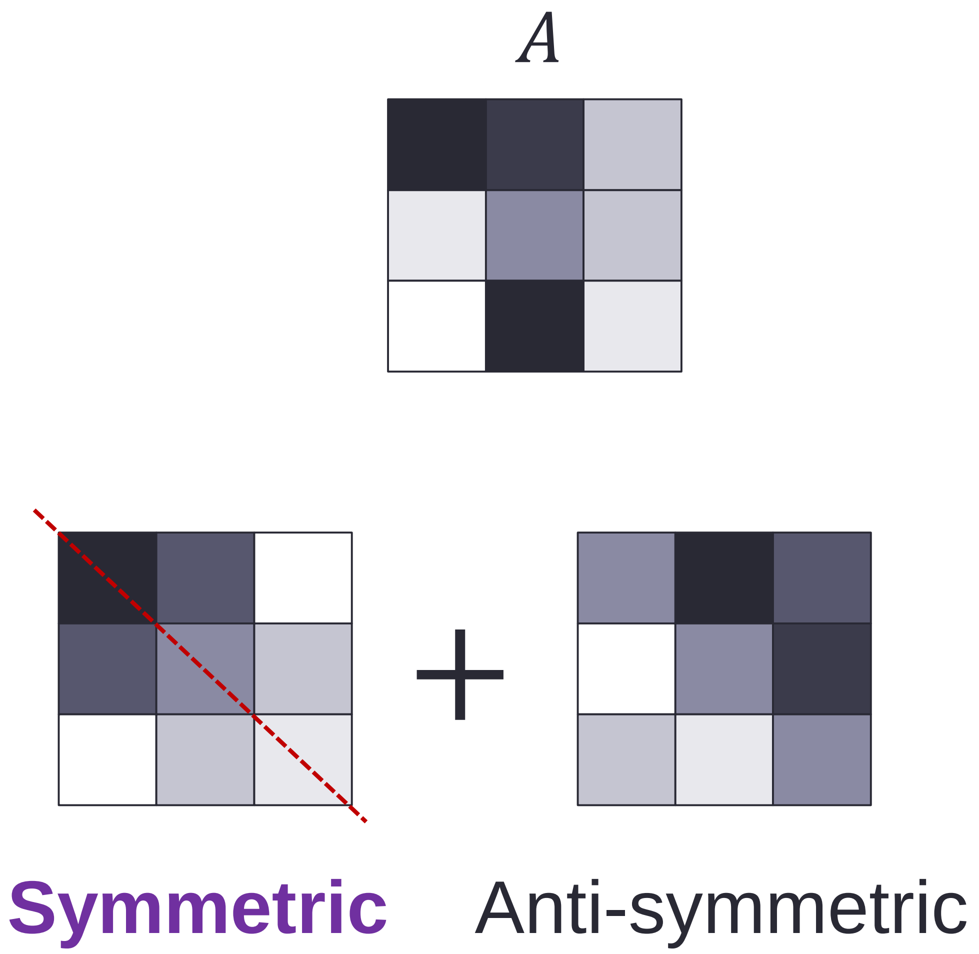
$$\gamma \leq \sigma_{\min}(A)$$

non-convex



cut a convex subset?

Bounding SVs from below

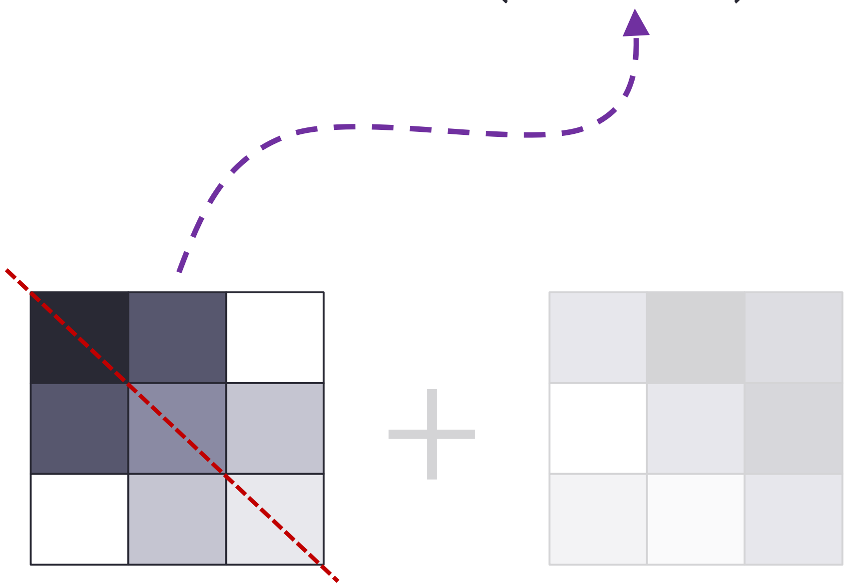


Bounding SVs from below

$$\gamma \leq \sigma_{\min}(A)$$

$\Leftarrow$

$$\gamma \leq \sigma_{\min}\left(\frac{A + A^T}{2}\right)$$

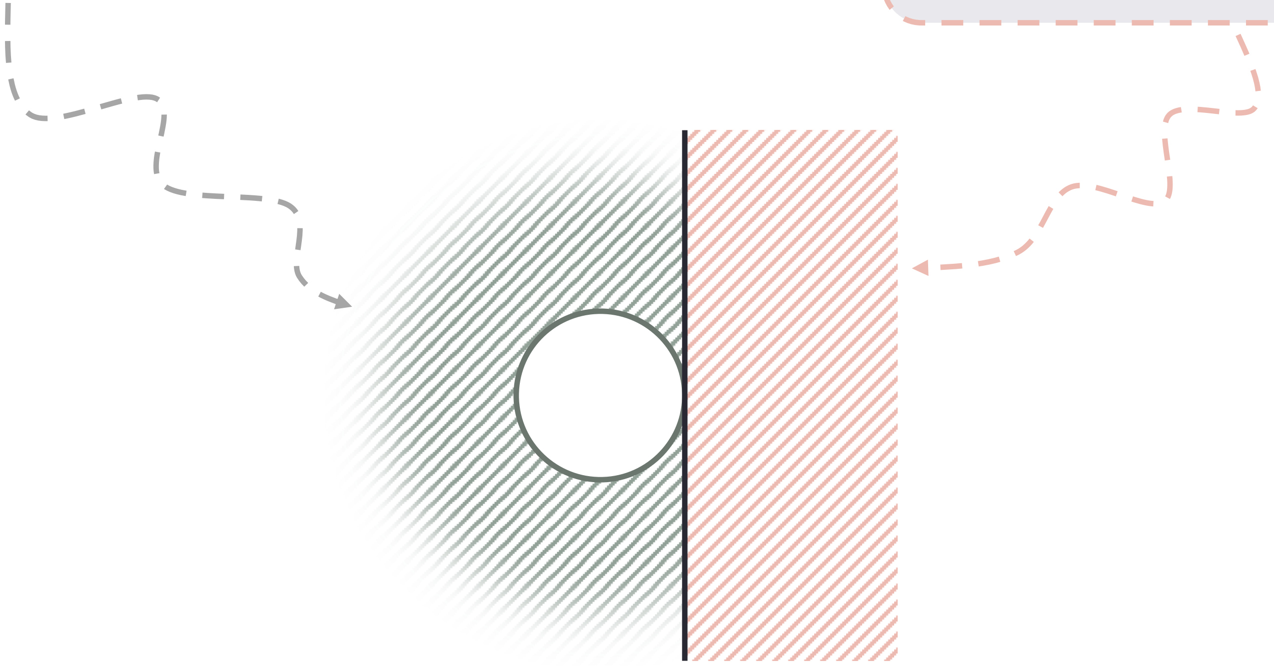


Bounding SVs from below

$$\gamma \leq \sigma_{\min}(A) \quad \Leftarrow \quad \gamma \leq \sigma_{\min}\left(\frac{A + A^T}{2}\right)$$

Bounding SVs from below

$$\gamma \le \sigma_{\min}(A) \quad \Longleftarrow \quad \frac{A + A^T}{2} - \gamma I \succcurlyeq 0$$

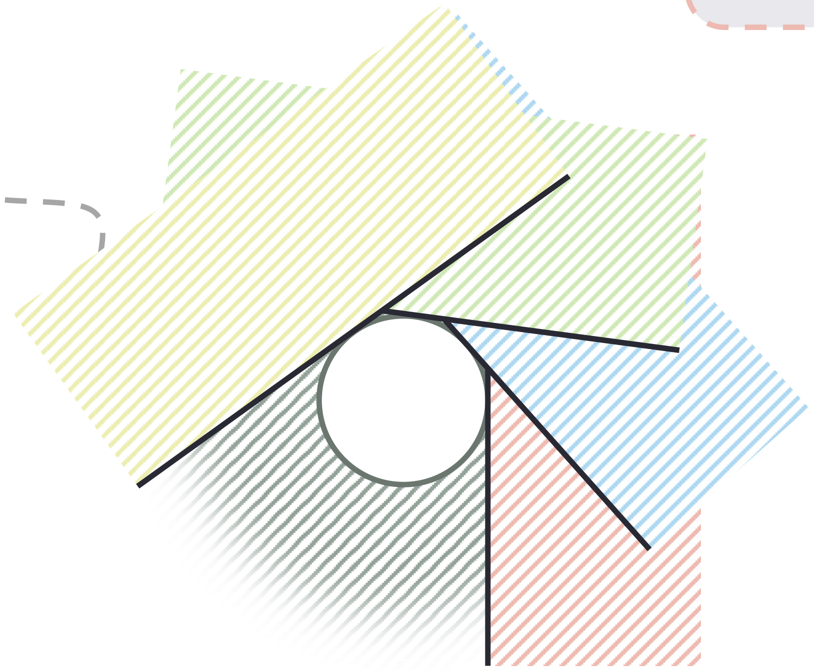


maximal convex subset

Maximal Convex Cover

$\gamma \leq \sigma_{\min}(A)$

$$\frac{A + A^T}{2} - \gamma I \succcurlyeq 0$$



maximal convex cover

Our Key Result

$$\begin{pmatrix} \Gamma I & A \\ A^T & \Gamma I \end{pmatrix} \succcurlyeq 0$$

$$\frac{A + A^T}{2} - 2\gamma I \succcurlyeq 0$$

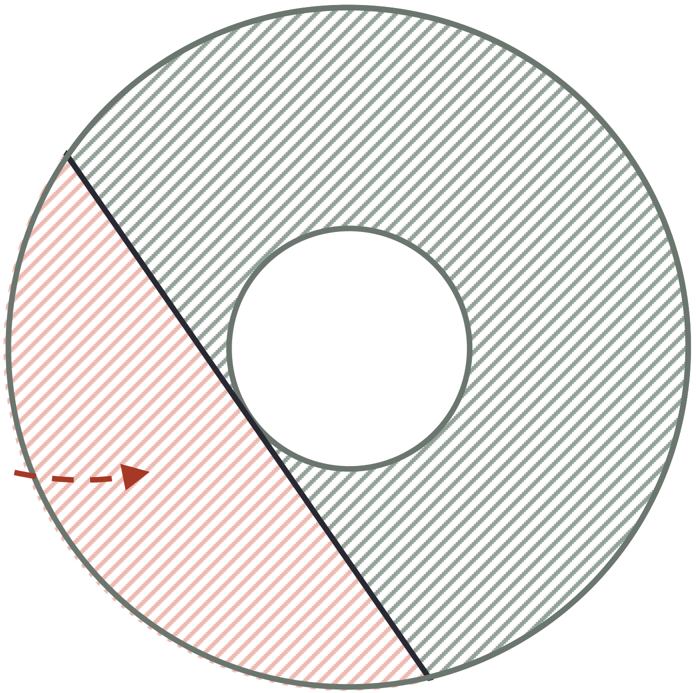
Convex-Optimal

maximal convex cover

SDP

Semidefinite Programming

$\gamma \leq \sigma(A) \leq \Gamma$



Our Key Result

$$\begin{pmatrix} \Gamma I & A \\ A^T & \Gamma I \end{pmatrix} \succcurlyeq 0$$

$$\frac{A + A^T}{2} - 2\gamma I \succcurlyeq$$

[Lipman 2012]

- Specific to 2D ⇒ SOCP

Ours

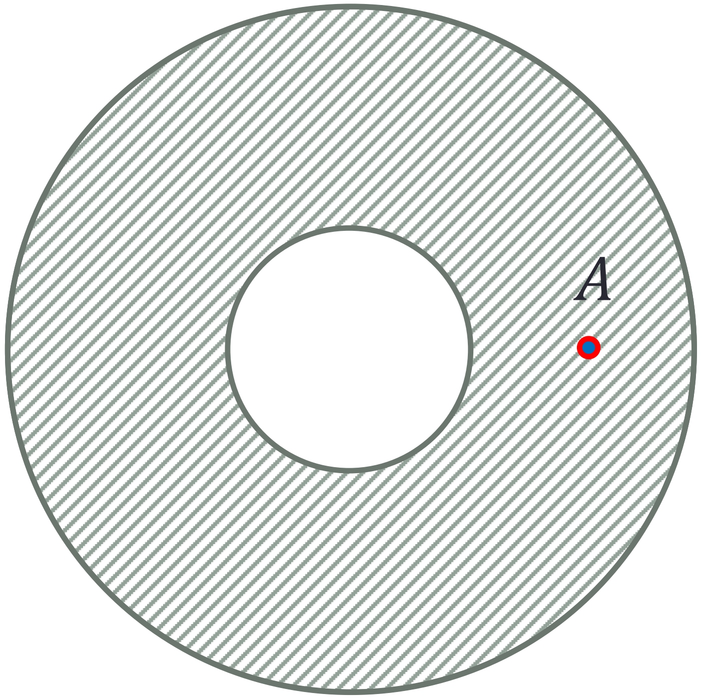
- Any dimension ⇒ SDP

Semidefinite Programming

(SDP)

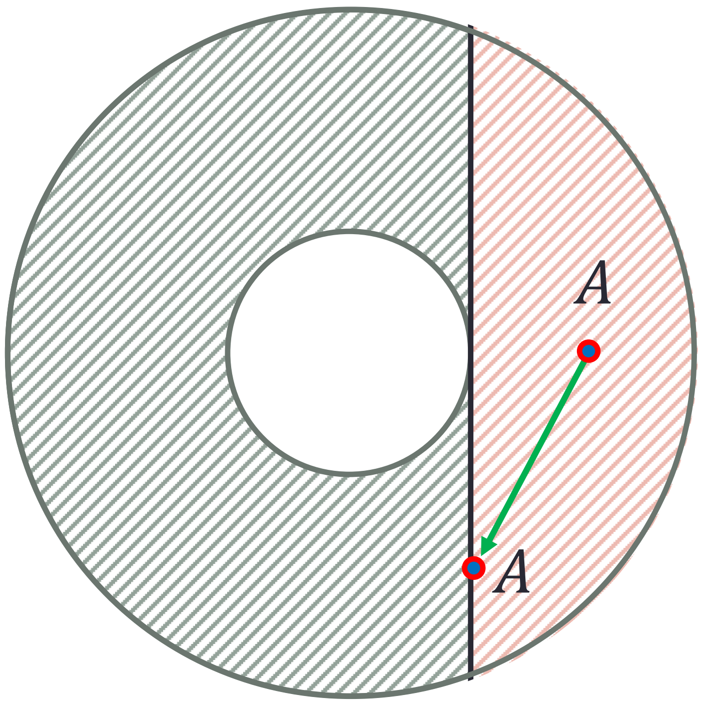
Optimization with SV constraints

- Start with feasible A



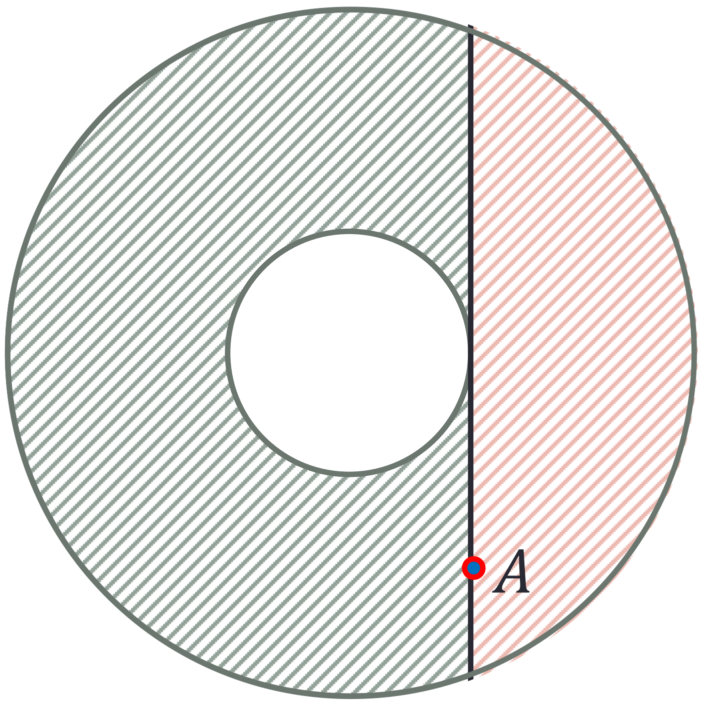
Optimization with SV constraints

- Start with feasible A
- Repeat:
  - Optimize over a maximal convex restriction (SDP)



Optimization with SV constraints

- Start with feasible A
- Repeat:
  - Optimize over a maximal convex restriction (SDP)
  - Update rotation (centering)



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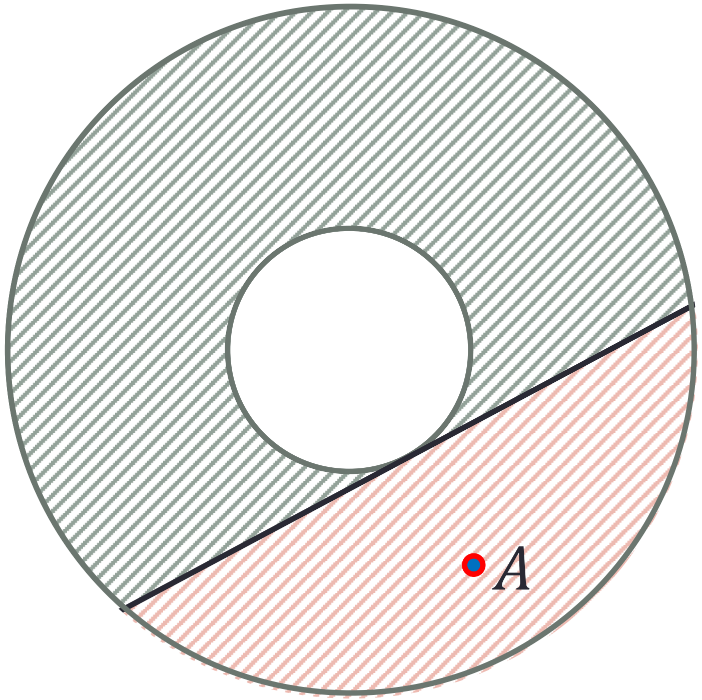

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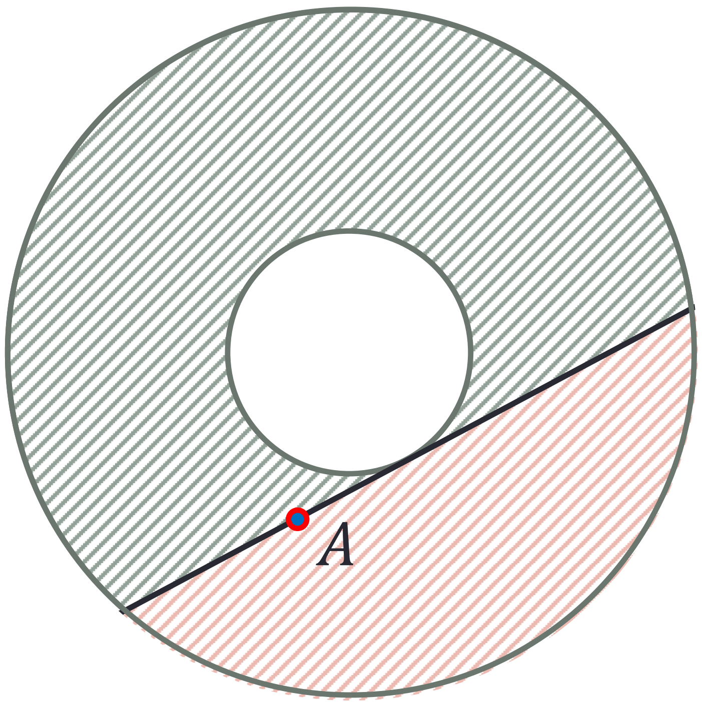
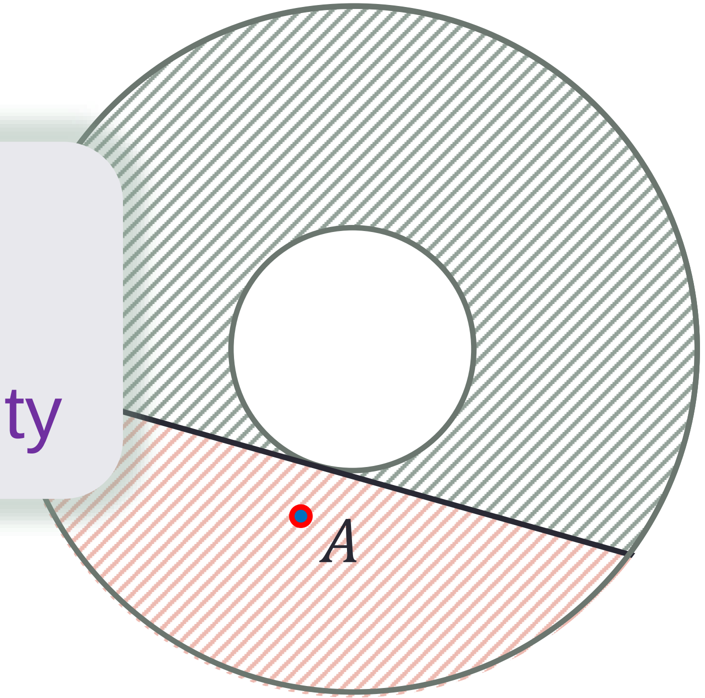


Optimization with SV constraints

- Start with feasible A
- Repeat:
  - Optimize over a maximal convex restriction (SDP)
  - Update rotation (centering)

Every iteration
energy decreases
guaranteed feasibility



Optimization with SV constraints

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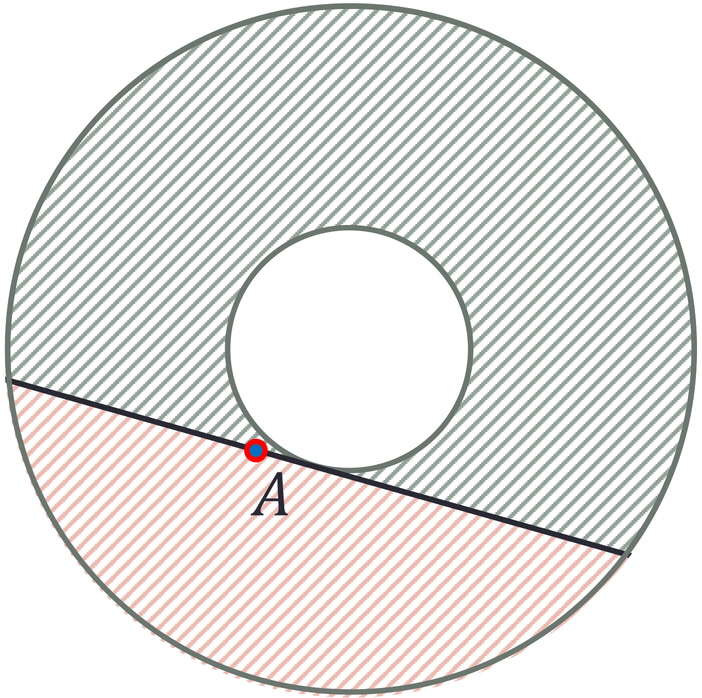

Optimization with SV constraints

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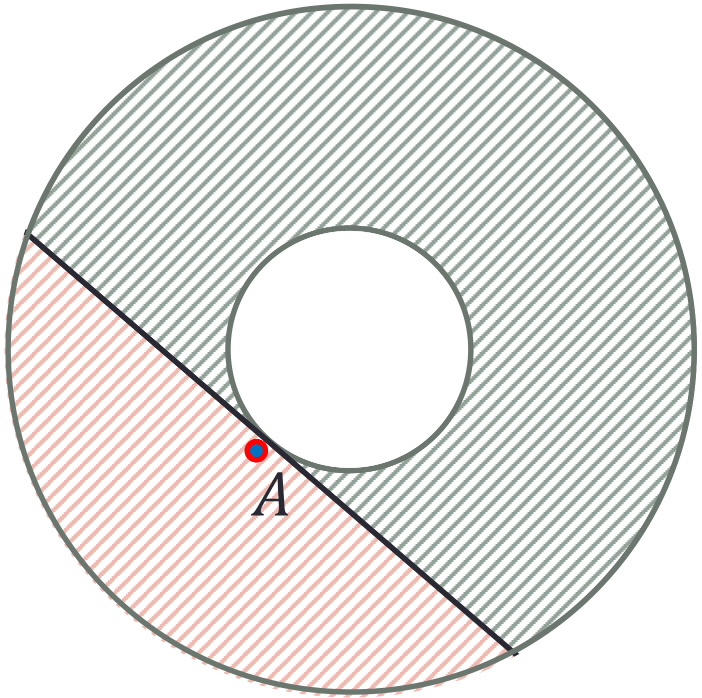

Optimization with SV constraints

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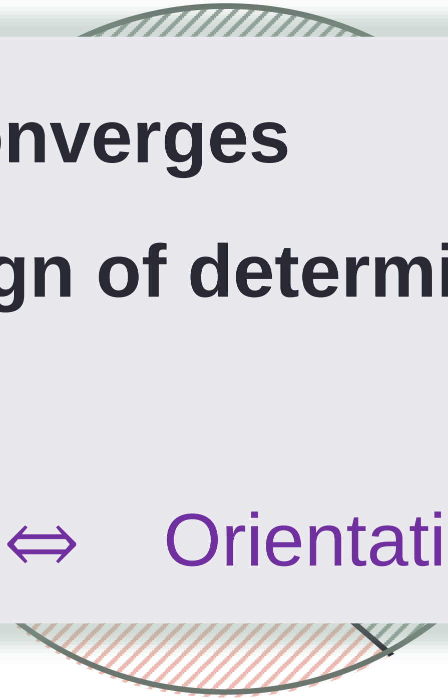


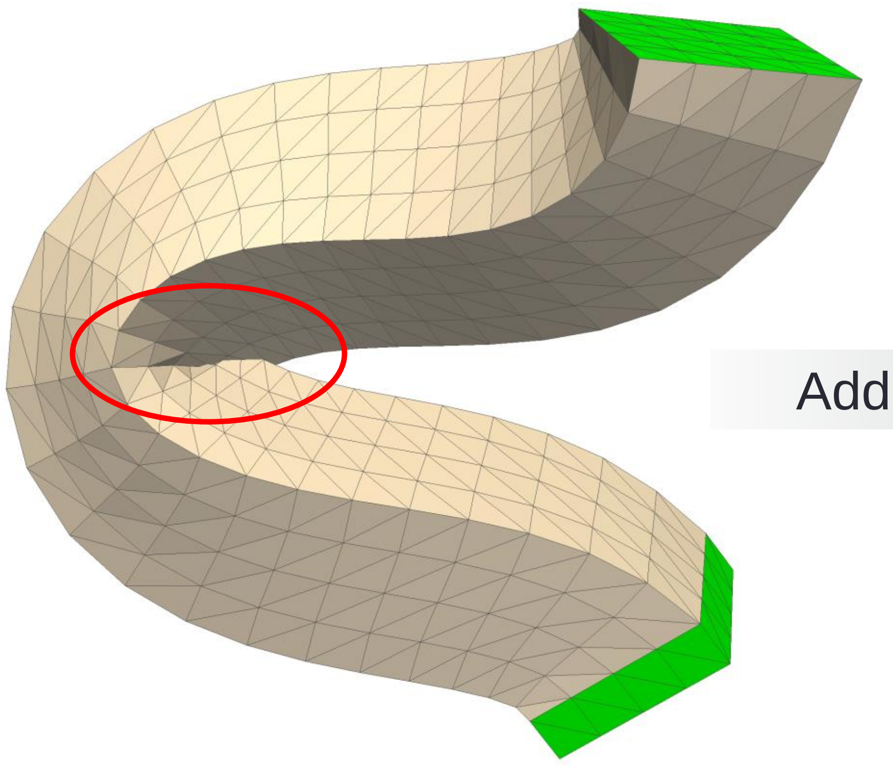
- **Algorithm converges**
- **Preserves sign of determinant**

$\det(A) > 0 \quad \Leftrightarrow \quad$ Orientation preserving

Applications

Shape Deformations

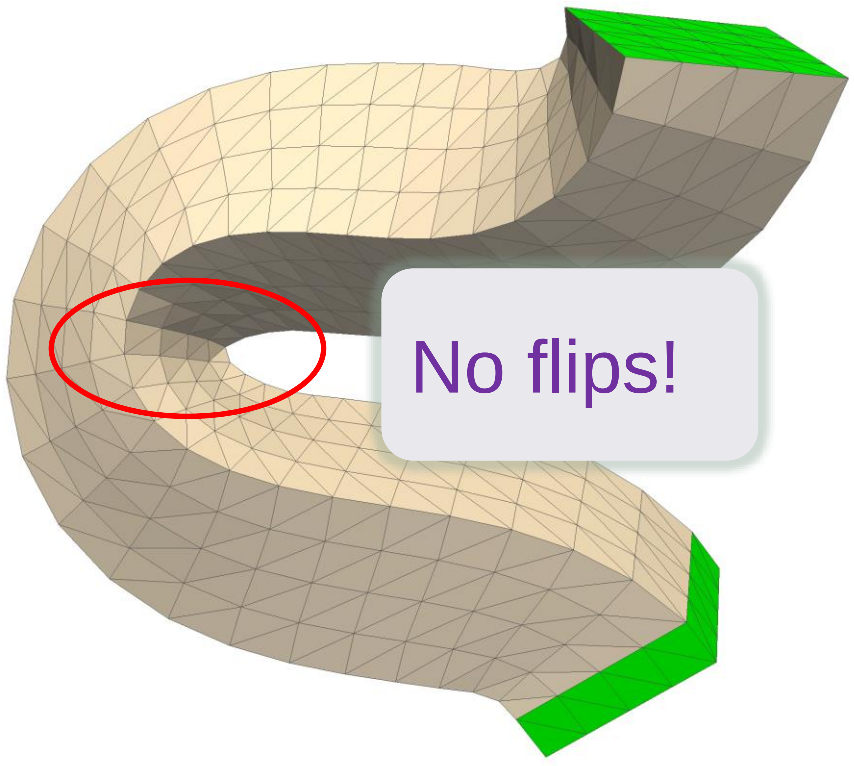
- As-Rigid-As-Possible [Sorkine2007]



Unconstrained

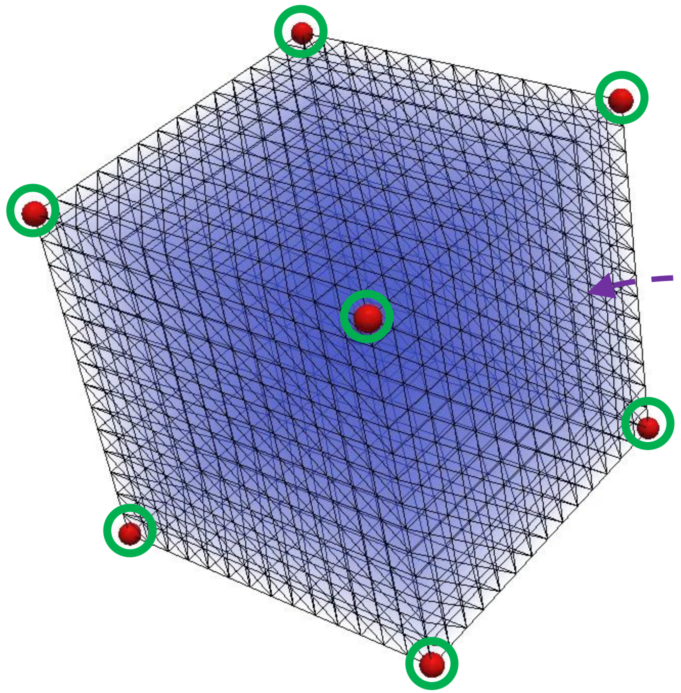
Add constraint on SVs

$$\frac{\sigma_{\max}(A_i)}{\sigma_{\min}(A_i)} \leq 2$$



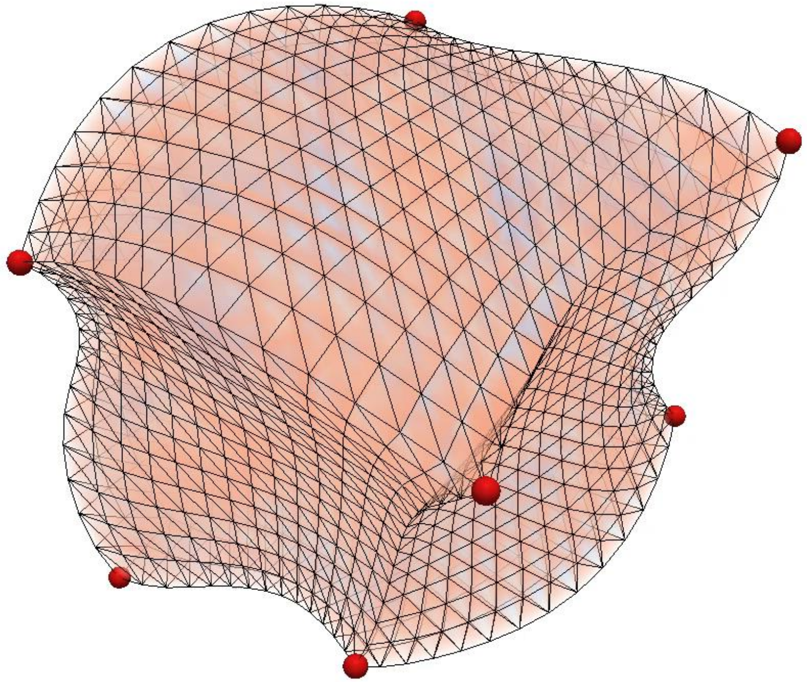
Bounded Conformal Distortion

Extremal Quasiconformal Mappings



$$\textbf{minimize}\left(\max_i \frac{\sigma_{\max}(A_i)}{\sigma_{\min}(A_i)}\right)$$

Extremal Quasiconformal Mappings

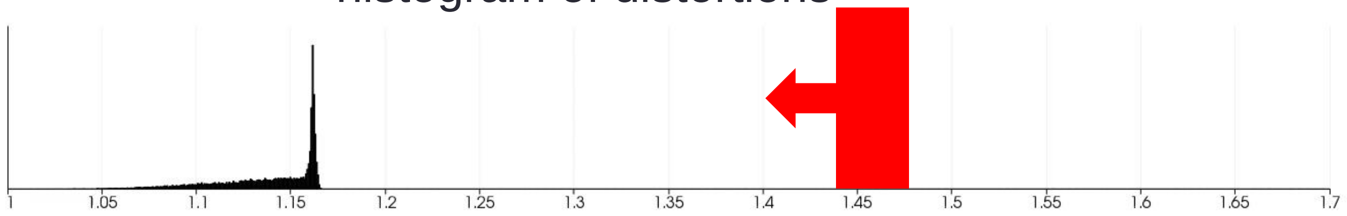


$$\textbf{minimize}\left(\max_i \frac{\sigma_{\max}(A_i)}{\sigma_{\min}(A_i)}\right)$$

"Most Conformal Mapping"

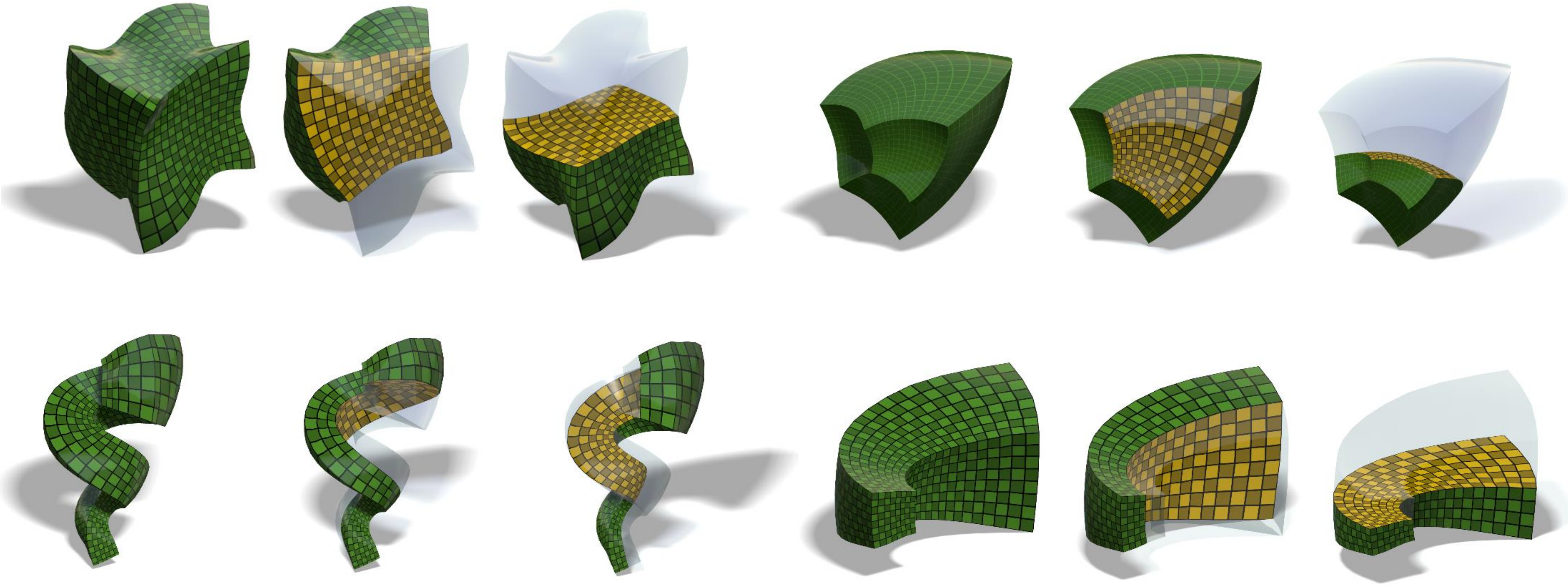
- Well studied in 2D [Weber et al. 2012]
- Little known in 3D…

histogram of distortions

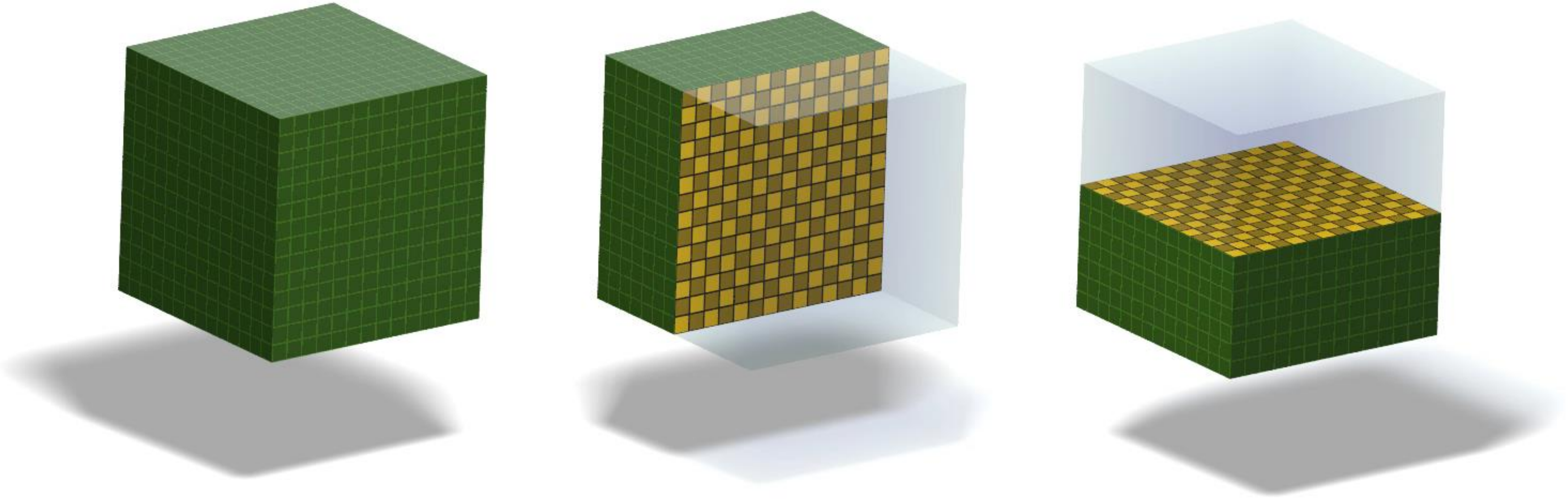


Extremal Quasiconformal Mappings

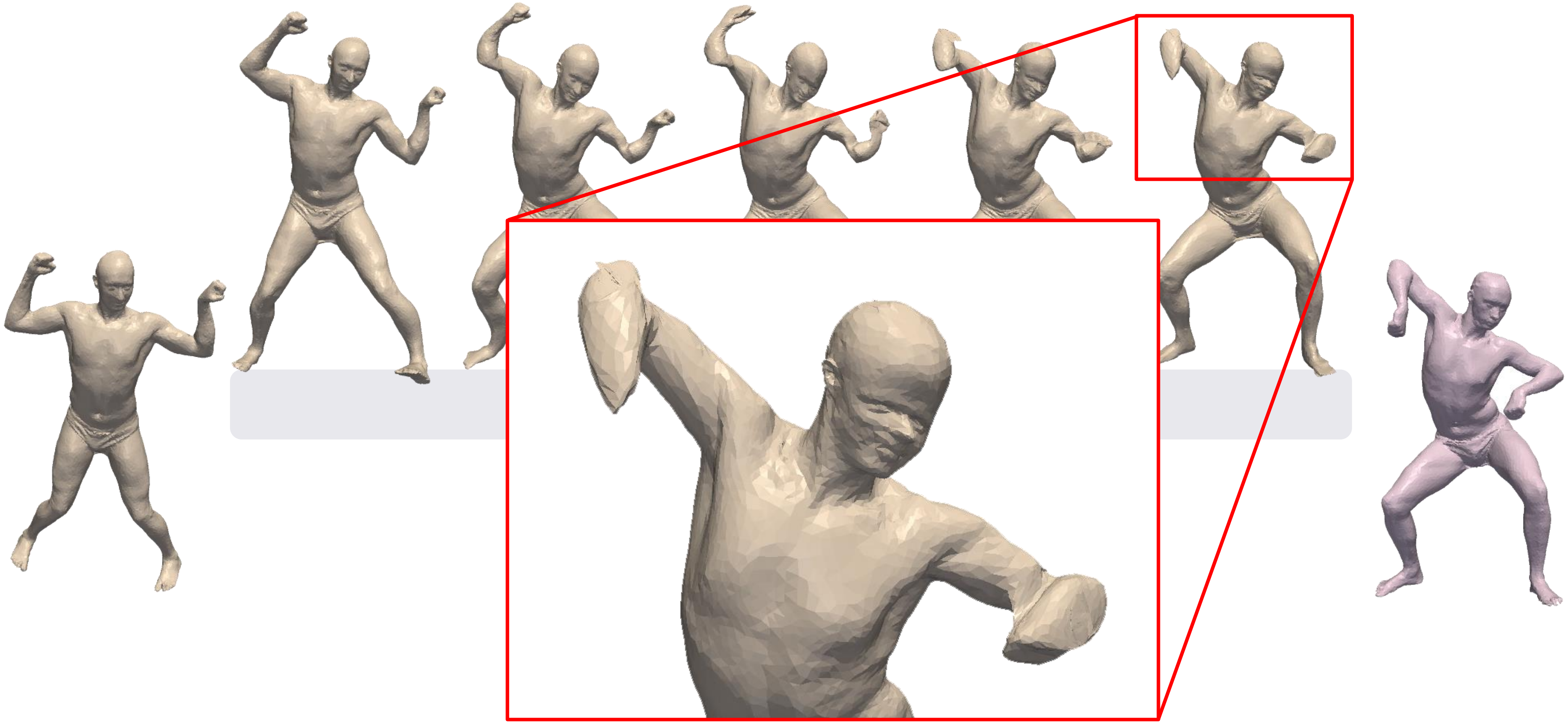
Approximate solution using our framework



Extremal Quasiconformal Mappings



Non-Rigid 3D ICP



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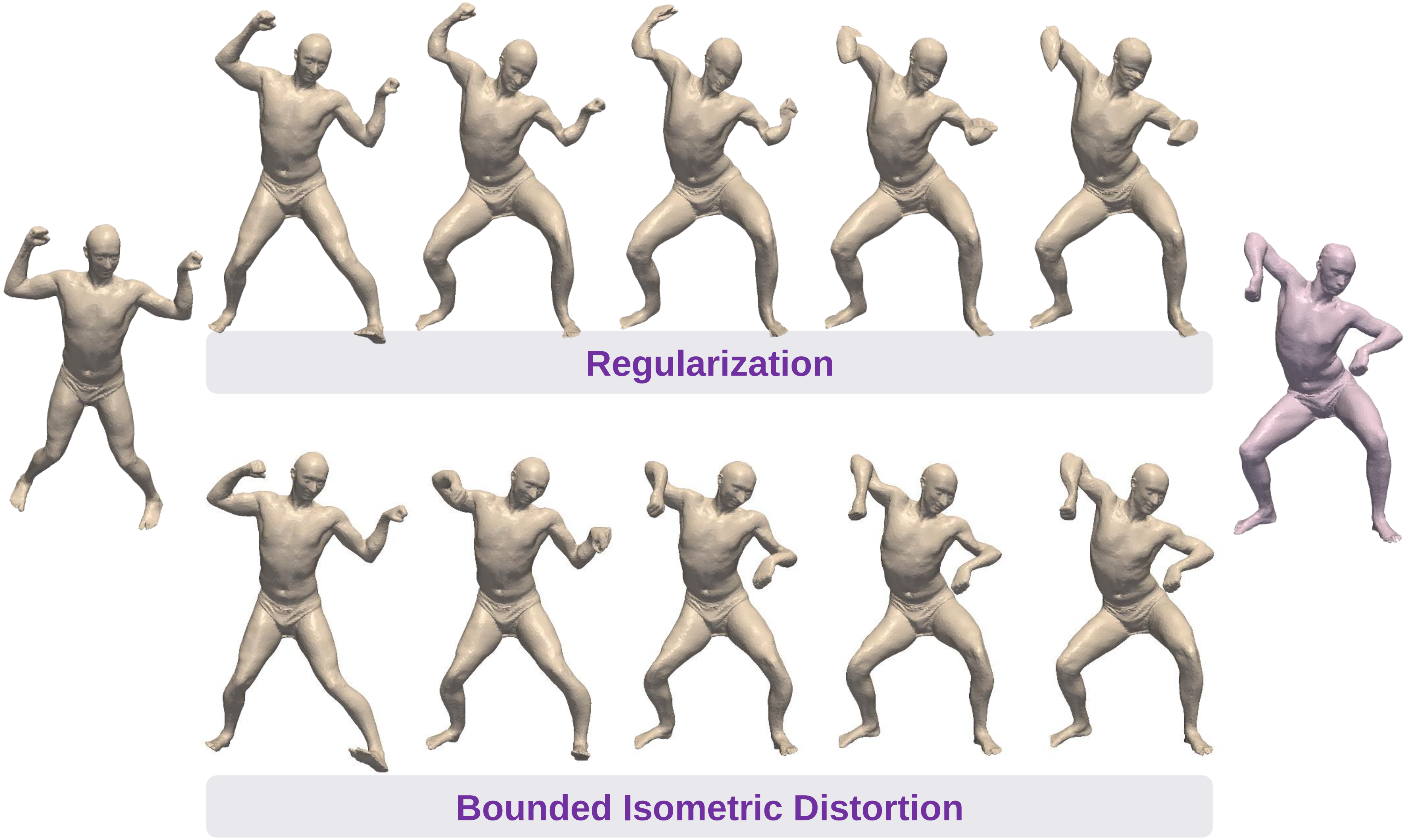
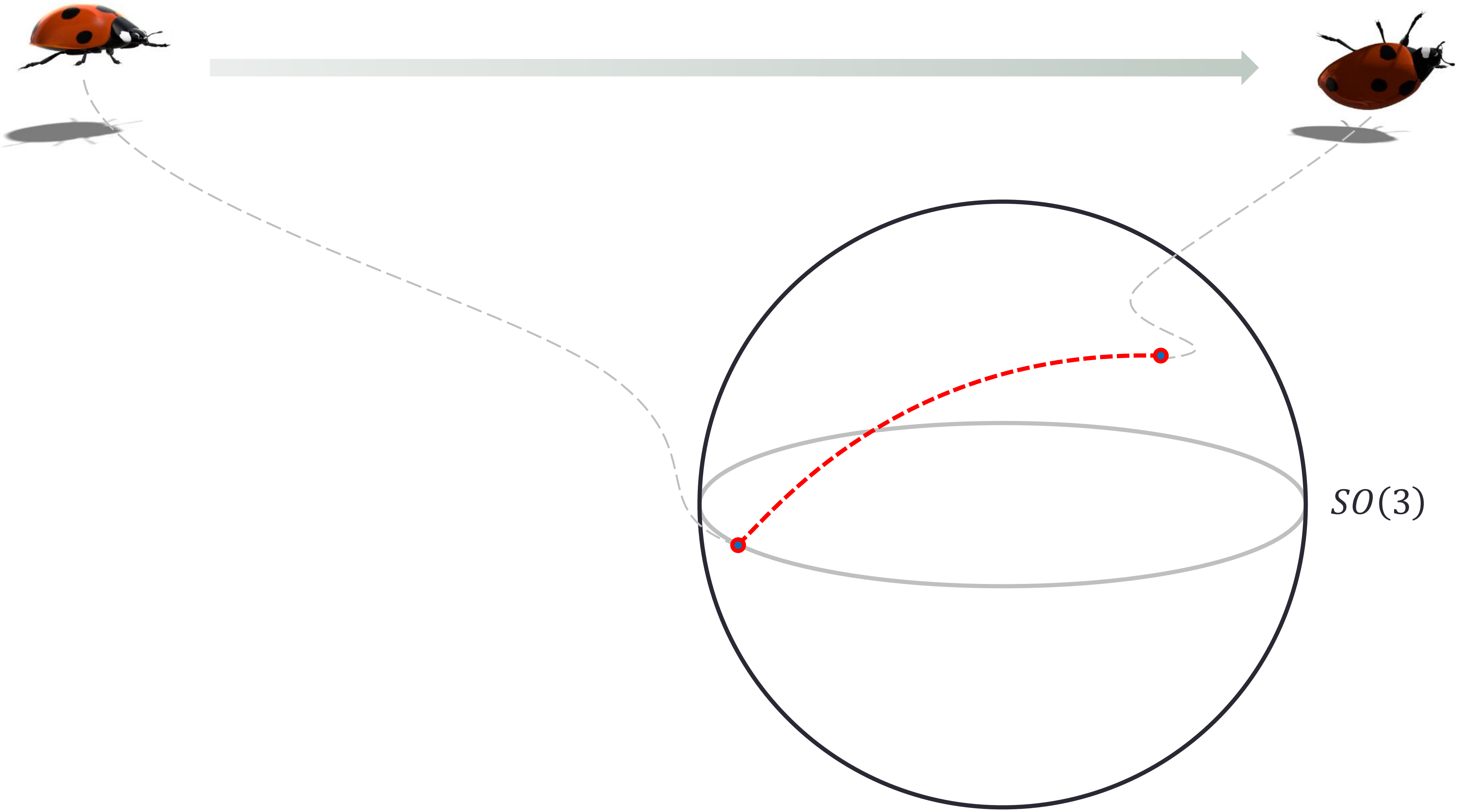


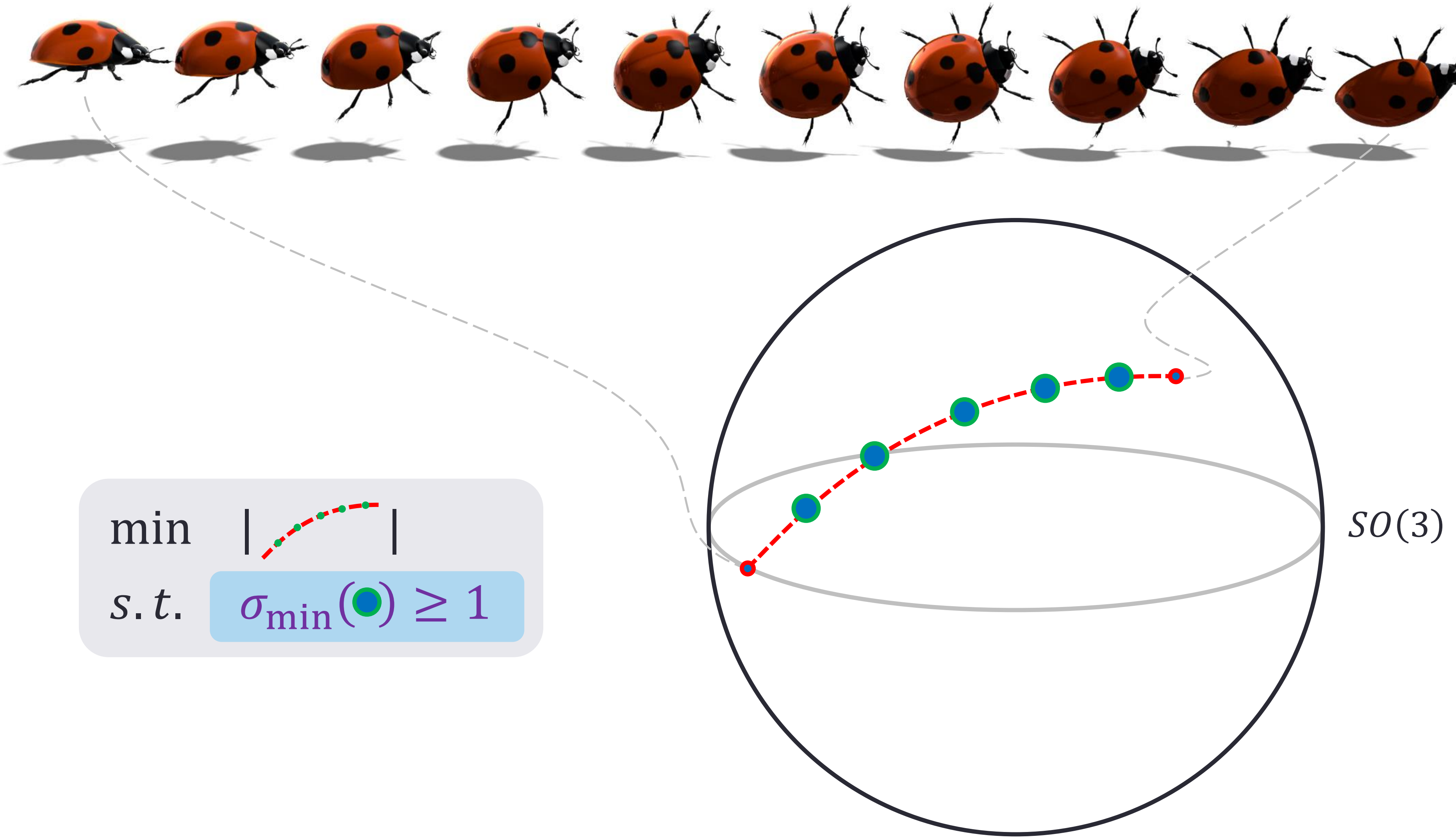
Non-Rigid 3D ICP

Non-Rigid 3D ICP

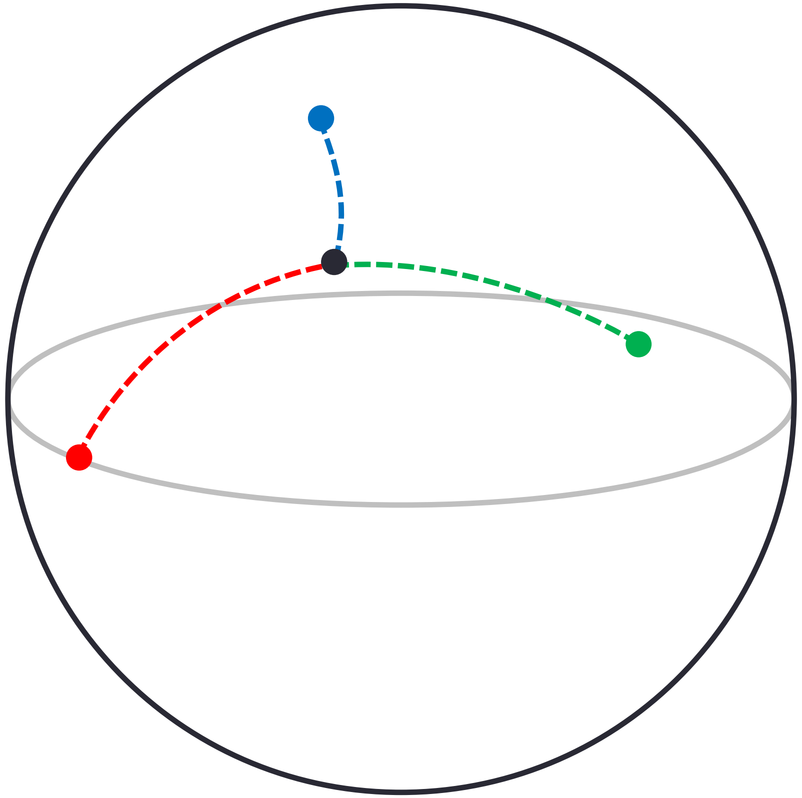
Interpolating Rotations



Interpolating Rotations



Averaging Rotations



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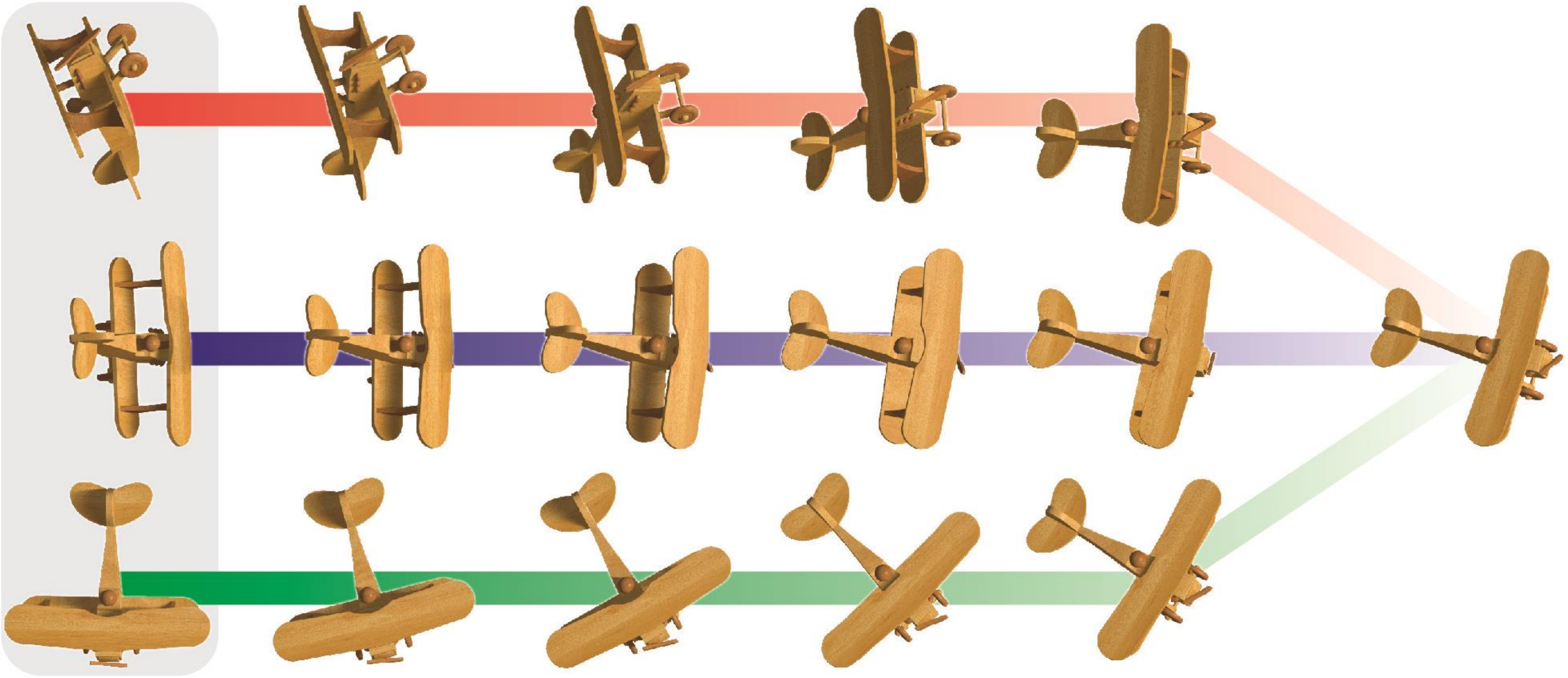
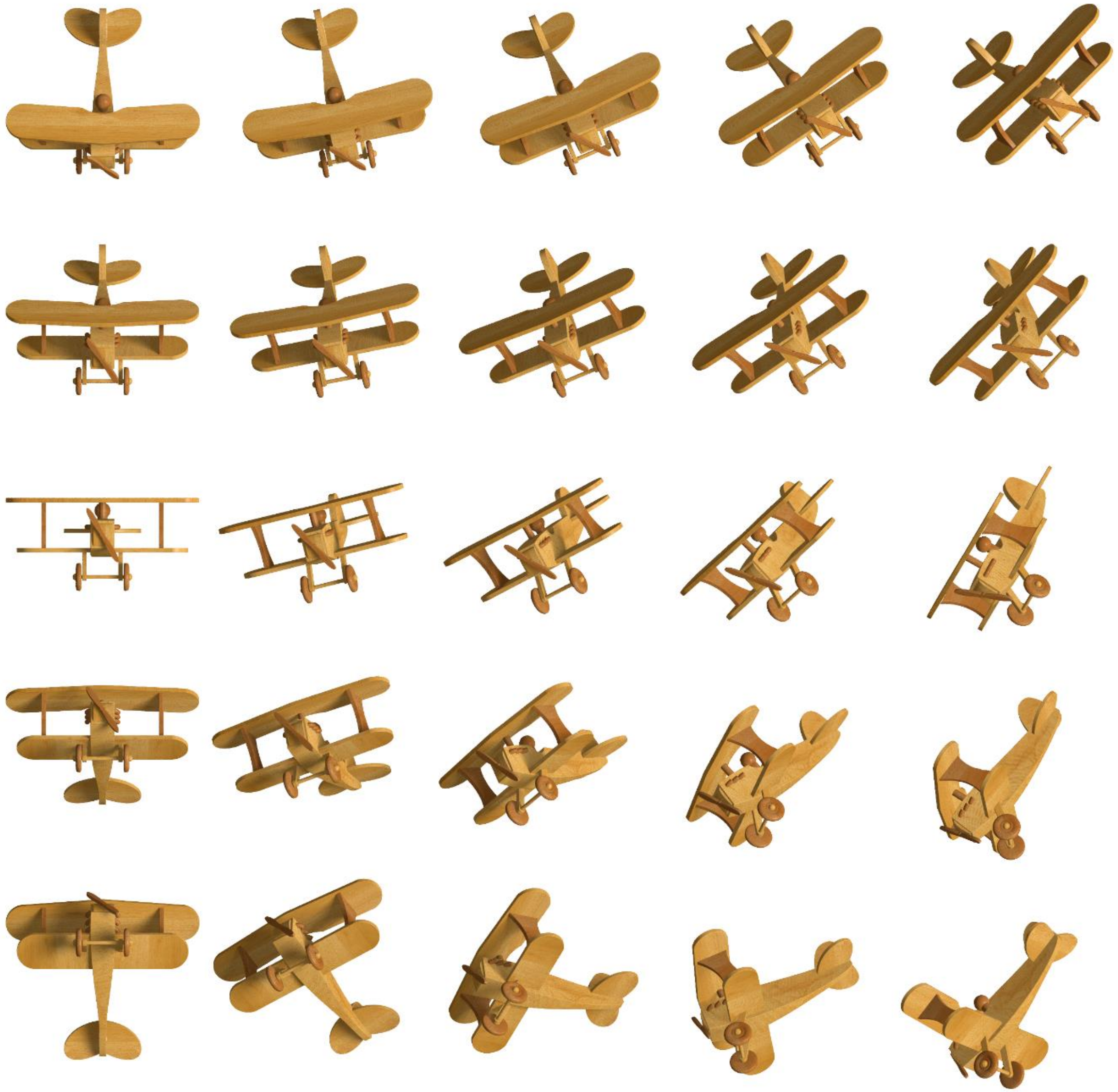


Interpolating Rotations



Concluding Remarks

- Method for optimizing

$$\min_{A \in \mathbb{R}^{n\times n}} \quad f(A, \sigma_{\min}(A), \sigma_{\max}(A))$$
$$s.t. \quad g_i(A, \sigma_{\min}(A), \sigma_{\max}(A)) \leq 0$$

- Based on SDP
- Optimal in the convex regime

- Main limitation: time complexity

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Thank you!