REGISTRATION OF JOINT GEOMETRIC AND RADIOMETRIC IMAGE DEFORMATIONS IN THE PRESENCE OF NOISE

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ABSTRACT

We consider the problem of object registration where the observed template simultaneously undergoes an affine transformation of coordinates and a non-linear mapping of the intensities. More generally, the problem is that of jointly estimating the geometric and radiometric deformations relating two observations on the same object. We show that, in the absence of noise, the original high dimensional non-convex search problem that needs to be solved in order to register the observation to the template is replaced by an equivalent problem, expressed in terms of a sequence of two linear systems of equations. A solution to this sequence provides an exact solution to the registration problem.

It is further shown that in the presence of noise, the original stochastic registration problem can be mapped, almost surely, to a new deterministic problem in the form of a classic deconvolution problem. Solution of the deconvolution problem reduces the solution of the original estimation problem to the form derived for the noise-free case.

Index Terms— Image registration, Image recognition, Parameter estimation, Nonlinear estimation, Multidimensional signal processing

1. INTRODUCTION

The problem of estimating radiometric and geometric deformations of observed objects is an elementary problem in computer vision. Its explicit, or implicit, solution is an essential part of any registration or recognition algorithm. Since pose, illumination and acquisition system vary, the set of possible observations on an object is immense. Thus, the task of determining the correspondence between two observations is extremely complicated.

We elaborate on the case where the global variability associated with the observation is both geometric and radiometric. Observations on an object are assumed to simultaneously undergo an affine transformation of coordinates and a non-linear mapping of the intensities.

In the geometric aspect, the case of affine transformations of coordinates is basic and provides a "first-order" approximation to more complex cases (such as "small" projective deformations, etc.). The radiometric aspect may be classified as a part of the general framework known as *color constancy* [1]. The type of global radiometric variability we consider is often referred to as "intensity mapping" or "camera response function"; it naturally appears in the important case of single-modal registration, where non-linearities are typically introduced (sometimes intentionally) by an image acquisition system as the overall non-linearity of its various components (CCD/CMOS, amplifiers, etc.) [2].

More specifically, let $g : \mathbb{R}^m \rightarrow \mathbb{R}$ be a function, representing a multi–dimensional signal (*e.g.*, $m = 2$ in the case where the observed signals are images). Consider an observation h on g of the form $h(\mathbf{x}) = Q(g(\mathcal{A}(\mathbf{x})))$, where Q is invertible and $\mathcal{A}$ is affine. The right-hand composition of g with $\mathcal{A}$ (composition from within) can be thought of as a spatial/time deformation (*i.e.*, a deformation of the coordinate system), while its left-hand composition with Q (composition from without) can be thought of as a memoryless non-linear input/output system applied to the amplitude of the signal. Hence, in image formation terminology, the physical model corresponding to such model is that of a simultaneous deformation of both geometry and radiometry.

From this point of view and in the absence of noise, given two functions (signals) g and h, the registration problem is then to find, if exists, a pair $(Q, \mathcal{A})$ such that $h(\mathbf{x}) = Q(g(\mathcal{A}(\mathbf{x})))$. Unfortunately, straightforward approaches for solving this problem typically lead to a high-dimensional non-convex search problem [3, 4, 5]. Hence, the direct approach is computationally demanding.

Image registration is a field of active research. In particular, various estimation methods have been proposed in the case of affine geometric transformations [3, 4, 6]. However, geometric registration methods that directly employ the intensity information of the image, typically fail in case there exists some non-negligible radiometric transformation relating the intensities in the template and the observation. This limitation leads, in turn, to the need to restrict the registration procedures to employ only a small fraction of the information available in the observations (which however is less sensitive to radiometric deformations) by considering edges, or feature landmark points. The problem of explicitly deriving a registration procedure in the case of combined radiometric and geometric variations has also gained some attention, for example see [5]. However, existing methods evade the inherent non-linearity of the this estimation problem through linear approximation and/or optimization.

We begin by showing that, in the absence of noise, the underlying algebraic structure of the problem and the notion of "empirical distribution" may be exploited to map the original highly-complicated joint estimation problem to an *equivalent* problem, expressed in terms of a sequence of two *linear* problems. Registration is then explicitly obtained by solving the two systems of linear equations.

In the second part, we assume that the observation is subject to an additive noise. We briefly present a framework of "uniform dense sampling" where the notion of "empirical distribution" is expanded to the noisy case. We show, that by solving a standard deconvolution problem, the solution of the original stochastic registration problem is reduced to the form derived in the noiseless case.

2. PROBLEM DESCRIPTION

Let $\mathcal{A} : \mathbb{R}^m \to \mathbb{R}^m$ be an affine transformation of coordinates, *i.e.*, $\mathcal{A} : \mathbf{x} \mapsto \mathbf{A}\mathbf{x} + \mathbf{c}$ where $\mathbf{A} \in \mathbb{R}^{m\times m}$ is non–singular and $\mathbf{c} \in \mathbb{R}^m$. $\mathcal{A}$ shall represent the geometric deformation. Let $Q : \mathbb{R} \to \mathbb{R}$ be an increasing invertible function, representing the nonlinear radiometric deformation. Let us further assume that $Q(0) = 0$.

Let $B_c^+(\mathbb{R}^m)$ denote the space of bounded, compactly supported, non-negative, Lebesgue measurable functions from $\mathbb{R}^m$ to $\mathbb{R}$. Let $g \in B_c^+(\mathbb{R}^m)$ and let $\{\eta(\mathbf{x}) : \mathbf{x} \in \mathbb{R}^m\}$ be a real-valued i.i.d. random field with a *known* distribution function F_η. The problem addressed in this paper is the following:

Given the known *function g and a single measurement (observation) h of the form*

$$h(\mathbf{x}) = [Q \circ g \circ \mathcal{A}](\mathbf{x}) + \eta(\mathbf{x}) \quad (1)$$
$$= Q(g(\mathcal{A}(\mathbf{x}))) + \eta(\mathbf{x})$$

for all $\mathbf{x} \in \operatorname{supp}\{Q \circ g \circ \mathcal{A}\}$*, find an* estimate *for the left-hand composition Q and the affine transformation* $\mathcal{A}$*, that is, the matrix* $\mathbf{A}$ *and the translation vector* $\mathbf{c}$.

Remarks. 1. $\operatorname{supp}\{f\}$ denotes the support the function f, (*i.e.*, the closure of the set where f does not vanish). The restriction $\mathbf{x} \in \operatorname{supp}\{Q \circ g \circ \mathcal{A}\}$ can be interpreted as the assumption of known segmentation.
2. An important special case is the Additive White Gaussian Noise (AWGN) model, where $\{\eta(\mathbf{x}) : \mathbf{x} \in \mathbb{R}^m\}$ is also assumed to be zero mean Gaussian with some *known* variance σ_η^2.

3. THE NOISELESS CASE: AN ALGORITHMIC SOLUTION

In this section we discuss the *noiseless* case, that is, where $\eta \equiv 0$ in (1). As we have thoroughly shown in [7], the problem can be solved in two stages: first, we use an affine-invariant transformation to isolate and estimate the left-hand composition Q; next, the solution for the function Q is used to reduce the original problem to a simpler one, where the parameters of the affine transformation, $\mathbf{A}$ and $\mathbf{c}$, are evaluated.

Denote by λ the Lebesgue measure on $\mathbb{R}^m$. Let us define the "empirical distribution" transformation T on $B_c^+(\mathbb{R}^m)$ by

$$[Ts](t) = \frac{1}{\lambda\{\operatorname{supp}\{s\}\}}\lambda\{\mathbf{x} \in \operatorname{supp}\{s\} : s(\mathbf{x}) \le t\} \quad (2)$$

for $s \in B_c^+(\mathbb{R}^m)$. The "empirical distribution" may be thought of as the "cumulative histogram" of a function. Notice the following properties of T:

Lemma 1. *[7], For a given* $s \in B_c^+(\mathbb{R}^m)$ *the following hold:*

(a) The function $S(t) = [Ts](t)$ *is a distribution function. Furthermore,*
  i. $S(t) = 0$ *for all* $t < 0$.
  ii. $S(t) = 1$ *for all* $t > \sup_x s(x)$.

(b) $T(s \circ \mathcal{A}) = Ts$.

(c) $T(W \circ s) = [Ts] \circ W^{-1}$ *for any strictly increasing continuous function* $W : \mathbb{R} \to \mathbb{R}$ *such that* $W(0) = 0$.

Applying T to relation (1) and using the above properties, we obtain the following functional relation (recall that $\eta \equiv 0$)

$$Th = T(Q \circ g \circ \mathcal{A}) = T(Q \circ g) = [Tg] \circ Q^{-1} \quad (3)$$

where to simplify the notation we omit the argument of the functions. In other words, Th and Tg are functionally related through a right-hand composition Q^{-1}. Hence, using the transformation T we have "converted" a functional relation expressed by a left-hand composition (*i.e.*, "radiometric deformation") into a new functional relation expressed by a right-hand composition (*i.e.*, "geometric deformation").

In fact, equation (3) describes a new, one-dimensional, "time-domain" registration problem. Hence, any suitable (parametric or non-parametric) registration method may be used to estimate Q^{-1}. In particular, in the case where Q is continuously differentiable, we have the following:

Let $\{e_i\}$ be a countable basis of $L_2(\operatorname{supp}\{Th\})$. By assumption, Q' is continuous, thus, is in $L_2(\operatorname{supp}\{Th\})$ and can be represented as

$$Q'(t) = \sum_i b_i e_i(t) \quad (4)$$

Using the estimation algorithm proposed in [8], any finite order model of the type (4) can be solved for the coefficients $\{b_i\}$ by means of solving a system of linear equations. Since $Q(0) = 0$, Q can be easily obtained by integration, which completes the estimation of Q.

Remark. Th and Tg are distribution functions, therefore, they are not compactly supported. This can be easily rectified by means of truncation [7], thus, the conditions of [8] are satisfied and the algorithm therein can be employed.

Having estimated Q, we can now estimate the geometric deformation: notice that (1) can be written as

$$h = [Q \circ g] \circ \mathcal{A} = \hat{g} \circ \mathcal{A} \quad (5)$$

where we define $\hat{g} = Q \circ g$. Since Q has been estimated and g is known, $\hat{g}$ represents a "new" template. Hence, (5) describes the relation of two known functions h and $\hat{g}$ related by an affine transformation of the coordinates.

That being the case, any suitable method for the registration of multi–dimensional affine transformations may be used. In particular, assuming g has no self affine symmetry [6], this equivalent problem can be easily solved for the unknown affine transformation $\mathcal{A}$ (*i.e.*, $\mathbf{A}$ and $\mathbf{c}$) using the algorithm proposed in [6], again by means of solving a low-dimensional system of linear equations.

Based on the conclusions in [6, 8], we conclude that if the derivative of Q admits a *finite* order representation in (4), and in the absence of noise, the overall solution for both the geometric and the radiometric deformations is completely determined and *exact*.

4. REGISTRATION OF NOISY OBSERVATIONS

In this section we discuss the *noisy* case, that is, where η in (1) is a real-valued i.i.d. random field with a *known* distribution function F_η. Due to space limitations the results are presented without proofs. Complete proofs can be found in [9].

Let us first informally discuss the case of noisy observations. The easiest way to get some intuition as to the notion of "empirical distribution" in the case of noisy observations is to consider for a moment the discrete case, where histograms exist. Next, we intuitively answer the following question: How does the histogram of the sum of an image and a noise process look like?

Each bin of the histogram represents the relative frequency of the respective level (value) in the image. Due to the influence of the additive noise, any image sample contributing to a single bin of the image histogram, say the n-th bin, will remain at the same level with some probability $p(\text{noise}=0)$. Similarly an image pixel originally contributing to the $n-1$ bin, may with probability $p(\text{noise}=1)$ contribute to the n-th bin of the noisy image histogram. Repeating the same argument for the entire support of the noise probability function, and summing-up all the contributions, we obtain the value of the n-th bin of the noisy image. Repeating the same argument for each bin of the image histogram implies that the histogram of the noisy observation is, qualitatively, the result of convolving the image histogram with the probability function of the noise.

In the following, we show that the notion of "empirical distribution" introduced in the previous section may be expanded to the case of noisy measurements, by formally expressing the above intuitive arguments.

Let us begin by introducing the following "discrete" transformations: let $\{u_i\}_{i=1}^{\infty}$ be a given sequence of points in $\mathbb{R}^m$; for any real-valued function s define the transformations $\{T_n^{\{u_i\}}\}_{n=1}^{\infty}$ by

$$\left[T_n^{\{u_i\}} s\right](t) = \frac{1}{n} \# \{i = 1, ..., n : s(u_i) \leq t\} \tag{6}$$

where $\#B$ denotes the cardinality of the set B. Whenever the limit $\lim_{n\to\infty} [T_n^{\{u_i\}} s](t)$ exists for all $t \in \mathbb{R}$, we define $T^{\{u_i\}}$ as

$$T^{\{u_i\}} s = \lim_{n\to\infty} T_n^{\{u_i\}} s \tag{7}$$

Remark. Notice that while the transformation $T_n^{\{u_i\}}$ is well defined for any real function, without a proper selection of a sequence $\{u_i\}_{i=1}^{\infty}$ the transformation $T^{\{u_i\}}$ is not necessarily defined.

To study the action of the above "discrete" transformations on deterministic functions we introduce some basic definitions and results from the theory of uniformly distributed sequences [10]. This theory is, in a sense, a theory of "uniform dense sampling". It asserts that the integral of "well-behaved" functions may be arbitrarily well approximated by sampling on an appropriate (countable) set of points. It is also shown that this set of "well-behaved" functions is exactly the set of Riemann integrable functions.

Closely related to Riemann's integral is the Jordan measure, defined as follows [11]: A (bounded) subset $U \subseteq \mathbb{R}^m$ is *Jordan measurable* if and only if the Lebesgue measure of its boundary vanishes, *i.e.*, $\lambda\{\partial U\} = 0$; or equivalently, if its indicator function χ_U is Riemann integrable. We can now define a uniformly distributed sequence:

Definition. Let $U \subseteq \mathbb{R}^m$ be a compact, Jordan measurable subset of $\mathbb{R}^m$ (of a positive measure). A sequence $\{u_i\}_{i=1}^{\infty} \subseteq U$ is said to be *uniformly distributed in U with respect to the Lebesgue measure* λ (abbreviated λ-u.d. in U) if

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} r(u_i) = \frac{1}{\lambda\{U\}} \int_U r(\mathbf{x}) d\lambda(\mathbf{x})$$

for every Riemann integrable function r with $\operatorname{supp}\{r\} \subseteq U$.

Remarks. 1. It can be shown that such sequences exist [9, 10]. In fact, they are natural in the sense that an independent sequence of realizations of a uniformly distributed random variable is almost surely λ-u.d.
2. The definition above *cannot* be generalized to Lebesgue measurable functions, as a Lebesgue integral cannot be evaluated on a countable set of points.

We shall now show the relationship between the transformation T and its discrete "version" $T^{\{u_i\}}$. In order to do so, we restrict the discussion to a better behaved class of functions. Define $\mathcal{R}_c^+(\mathbb{R}^m)$ to be the set of functions in $B_c^+(\mathbb{R}^m)$ that also are Riemann integrable. Restricting the discussion to $\mathcal{R}_c^+(\mathbb{R}^m)$ imposes no significant practical limitations as it is "rich" enough to describe any sampled physical signal. In this setting, we have the following result to tie up the "discrete" distribution transformation with the "continuous" one:

Lemma 2. *Let $U \subseteq \mathbb{R}^m$ be a compact, Jordan measurable subset of $\mathbb{R}^m$ and let $\{u_i\}_{i=1}^{\infty}$ be a λ-u.d. sequence in U. For all $r \in \mathcal{R}_c^+(\mathbb{R}^m)$ with* $\operatorname{supp}\{r\} = U$ *we have*

$$Tr = T^{\{u_i\}} r$$

If, in addition, r admits finitely many values, then for all t we have

$$\frac{\lambda\{\mathbf{x} \in U : r(\mathbf{x}) = t\}}{\lambda\{U\}} = \lim_{n\to\infty} \frac{\#\{i = 1, \ldots, n : r(u_i) = t\}}{n}$$

Hence, for a proper selection of sequence $\{u_i\}_{i=1}^{\infty}$ and on the well-behaved class of functions $\mathcal{R}_c^+(\mathbb{R}^m)$, the transformation T can be calculated by means of $\{T_n^{\{u_i\}}\}_{n=1}^{\infty}$.

Until now, we have discussed the application of "empirical distribution" transformations to deterministic functions. We shall now discuss the results of applying the transformations $T_n^{\{u_i\}}$ and $T^{\{u_i\}}$ to a random field.

Recall that $\{\eta(\mathbf{x}) : \mathbf{x} \in \mathbb{R}^m\}$ is a real-valued i.i.d. random field on $(\Omega, \mathcal{F}, P)$ with a *known* probability distribution function F_η. For a given sequence $\{u_i\}_{i=1}^{\infty}$ of *distinct* points in $\mathbb{R}^m$, the transformation $T_n^{\{u_i\}}$ can now be applied to η. In this context, we can rephrase the *Glivenko-Cantelli* theorem [12]:

$$\lim_{n\to\infty} \left[T_n^{\{u_i\}} \eta\right](t) = F_\eta(t) \quad \text{a.s. uniformly in } t.$$

Therefore, in terms of the transformations we have previously defined, $T^{\{u_i\}}\eta = F_\eta$ with probability 1. Hence, for *any* sequence of distinct points $\{u_i\}_{i=1}^{\infty} \subseteq \mathbb{R}^m$ the transformation $T^{\{u_i\}}$ is a *strongly consistent non-parametric estimator* for the probability distribution function of the random field $\{\eta(\mathbf{x}) : \mathbf{x} \in \mathbb{R}^m\}$.

We now return to discuss the problem stated in (1) and derive our main results. Recall that one is given single measurement (observation) of the form (1). Further assume that $g \in \mathcal{R}_c^+(\mathbb{R}^m)$, *i.e.*, g is a bounded, compactly supported, non-negative, Riemann integrable function from $\mathbb{R}^m$ to $\mathbb{R}$. Denote $\tilde{g} = Q \circ g \circ \mathcal{A}$. Obviously $\tilde{g} \in \mathcal{R}_c^+(\mathbb{R}^m)$ as well. Let $U = \operatorname{supp}\{g\}$ and $\tilde{U} = \operatorname{supp}\{\tilde{g}\} = \mathcal{A}^{-1}(U)$.

Let $\{u_i\}_{i=1}^{\infty}$ and $\{\tilde{u}_i\}_{i=1}^{\infty}$ be λ-u.d. sequences of distinct points in U and $\tilde{U}$ respectively.

Proposition 1. *If g admits finitely many values, the following holds*

$$\left[T^{\{\tilde{u}_i\}} h\right](t) = \sum_{r=1}^{R} \frac{\lambda\left\{\mathbf{x} \in \tilde{U} : \tilde{g}(\mathbf{x}) = \tilde{v}_r\right\}}{\lambda\left\{\tilde{U}\right\}} F_\eta(t - \tilde{v}_r)$$

almost surely, where $\{\tilde{v}_1, \ldots, \tilde{v}_R\}$ is the range of $\tilde{g}$.

If we further assume that the random field $\{\eta(\mathbf{x}) : \mathbf{x} \in \mathbb{R}^m\}$ has an absolutely continuous probability distribution function we have the following result:

Proposition 2. *Let f_η be the probability density function of the random field $\{\eta(\mathbf{x}) : \mathbf{x} \in \mathbb{R}^m\}$, then*

$$T^{\{\tilde{u}_i\}} h = \left(\left[T^{\{u_i\}} g\right] \circ Q^{-1}\right) * f_\eta \quad \textit{a.s.} \tag{8}$$

Discussion. The stochastic relation expressed in (1) describes a noisy observation on a template deformed both geometrically and radiometrically. Using the transformation $T^{\{u_i\}}$ we have (almost surely) "converted" this stochastic relation to a new *deterministic* functional relation. This relation is expressed by a right-hand composition Q^{-1} (*i.e.*, a deformation of coordinates), followed by the action of an LTI system. Interestingly, the impulse response of this LTI system is given by f_η, which represents the statistics of the random field η.

Hence, in order to estimate the left-hand composition Q, the original stochastic registration problem can be replaced, with probability one, by a "new" deterministic problem. This deterministic problem has the form of a "classic" deconvolution problem. Solution of the deconvolution problem reduces (8) to the form (3) derived for the noise-free case. Thus, as in the noiseless case, the original problem can then be solved in two stages using [6, 8].

Also note that (8) implies robustness of the noiseless algorithm when applied to a noisy measurement of high signal to noise ratio. The high SNR case can be characterized by a probability density function f_η that approximates the Dirac delta function. Thus, in high SNR, (8) naturally reduces to the noiseless equation (3), and Q may be estimated as before, as illustrated in the next section.

5. CONCLUDING NUMERICAL EXAMPLE

5.1. The Noiseless Case

Let us first illustrate the proposed method in the noiseless case, described in Section 3. In this example, the template image g would be the two-dimensional RGB image of a "flaming fractal" shown in Figure 1. The template is of dimension 1188×891 and the values of each channel are in the range of $[0, 1]$.

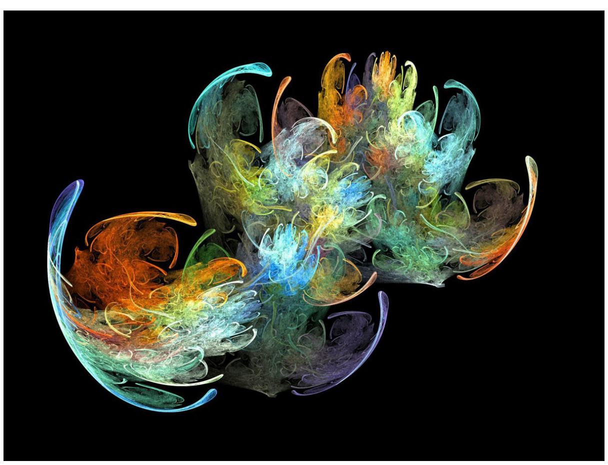

Fig. 1. The template image g.

The observed deformed image h is shown in Figure 2. It is a version of the template subject to both geometric and radiometric deformations. The geometric deformation is an affine transformation of coordinates (without translation). The radiometric deformation is due to three different point-wise nonlinear mappings applied to the intensities (amplitudes) of each of the template's channels. The mappings Q_R, Q_G and Q_B, respectively corresponding to the RGB channels of the image, were chosen to be the following polynomials

$$Q_R(t) = 2t - t^2; Q_G(t) = 2t^2 - t^4; Q_B(t) = t$$

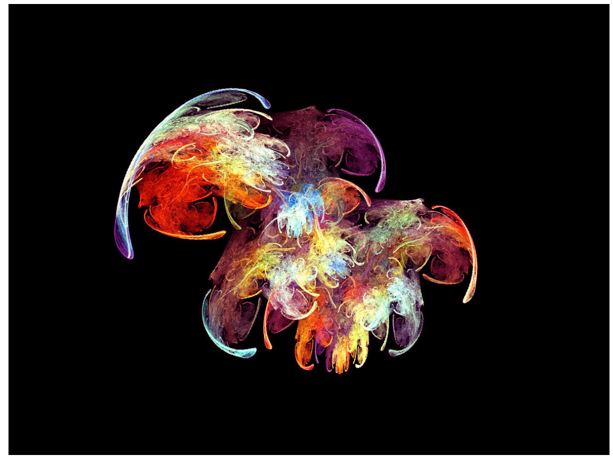

Fig. 2. The deformed observation image h.

The significant difference between the template and the observation images, both geometrically and radiometrically (color-wise), is easily noticeable. Therefore, intensity based geometric registration methods will typically fail due to the non-linear mapping of intensities. On the other hand, salient feature based methods (landmark methods), which are in a sense invariant to global mappings of intensities, will have "little to grasp on" in our example due to the "fractal" nature of the images, as emphasized by the example above.

Q_R, Q_G and Q_B were individually estimated by applying the proposed algorithm on each of the image channels. The intensity mapping functions and their corresponding estimates, as obtained by the proposed algorithm, are shown in Figure 3. The estimation errors are $2.67 \cdot 10^{-7}$; $9.02 \cdot 10^{-8}$ and $1.95 \cdot 10^{-7}$, respectively, (evaluated as the L_2 norm of the difference between the correct and estimated functions, normalized by the integration interval).

The affine deformation of coordinates is given by

$$\mathbf{A} = \begin{bmatrix} 1.3 & 0.3 \\ 0.1 & -1.3 \end{bmatrix}$$

The estimate is obtained jointly for all channels by stacking the sets of equations obtained using the individual channel information (see [6]) into a single over–determined system. This system is solved for the elements of A using a least squares solution. This estimate yields

$$\hat{\mathbf{A}} = \begin{bmatrix} 1.2970 & 0.2908 \\ 0.0980 & -1.3009 \end{bmatrix}$$

The error in the estimation of $\mathbf{A}$ is $5.77 \cdot 10^{-5}$, (where we take the estimation error to be $\|I - A^{-1}\hat{A}\|_2^2$).

5.2. The Case of Noisy Observations

In this subsection, we examine the performance of the algorithm in the case of noisy observations of high SNR. As pointed out at the end of Section 4, the derivation and results in the case of noisy observations imply the robustness of the (computationally non–demanding)

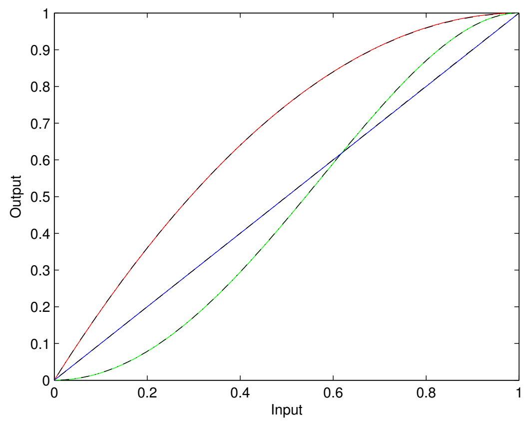

Fig. 3. The radiometric deformations Q_R, Q_G and Q_B of the three image color channels (red, green and blue solid lines) and their estimates (dashed lines).

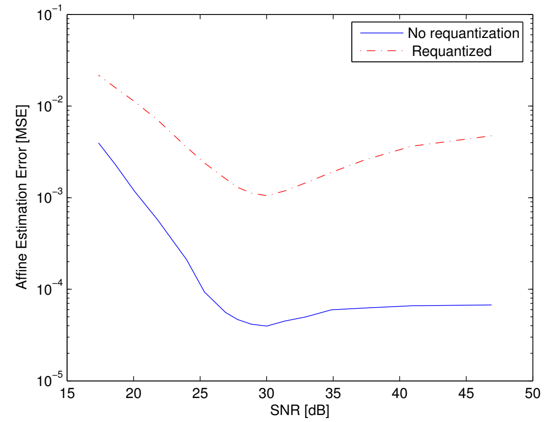

Fig. 4. The error in the estimation of the affine deformation $\mathcal{A}$.

noiseless algorithm when applied to a noisy measurement of high signal to noise ratio.

Sequences of Monte Carlo experiments were performed at various SNRs ($20dB \div 45dB$). The observations h were generated as before, and a zero mean white Gaussian noise with the appropriate variance was added.

Remarks. 1. On an image with 256 intensity levels (8 bits) per channel, the additive white Gaussian noise was chosen to have an STD equivalent to $0.5 \div 15$ levels. In experiments, a consumer digital camera (NIKON 8700), in standard room lighting conditions, has exhibited an inherent noise with an STD equivalent to $0.5 \div 2$ levels. **2.** The SNR was calculated as the RMS SNR across the 3 color channels.

As before Q_R, Q_G and Q_B were individually estimated by applying the proposed algorithm on each of the image channels. As expected, in lower SNR the mismatch of the noiseless algorithm is no longer negligible, and a slight bias in the estimation of Q_R, Q_G and Q_B is introduced. Nevertheless, estimation errors (MSE) were quite small ($\approx 10^{-5}$ in the low SNR experiments). It should be noted that the estimation variance was practically zero, in correspondence with the strong consistency of the left hand composition estimation procedure, expressed in (8); hence, the MSE is mostly comprised of a deterministic bias, due to the mismatch of the noiseless algorithm.

Next, the estimation procedure for the affine deformation $\mathbf{A}$ was applied. The estimation errors (MSE, in the sense previously defined) are shown in Figure 4. As before, the estimation variance was practically zero; hence, again the MSE is mostly comprised of a deterministic bias, due to the imperfect estimation of the radiometric deformation.

As a concluding experiment, we repeated the Monte Carlo sequence. This time, in a more "realistic" setting: we requantized the observation image h, to 256 uniformly spaced levels. The estimation results of the affine deformation are also shown in Figure 4. As expected, results are slightly worse than in the continuous (unquantized) case, yet not significantly.

Interestingly, somewhat better results are exhibited at SNR$\approx 30dB$. This can be informally explained as follows: the entire estimation method strongly depend on the "empirical distribution". Obviously, this distribution is severely corrupted by quantization. In a sense, the perturbations introduced by the noise compensate for the loss of information caused by the requantization, leading to a better approximation of Th, the "empirical distribution" of h. Hence, at certain SNRs, the overall estimation performance is improved. Further discussion of the requantization effects is out of scope of this paper.

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