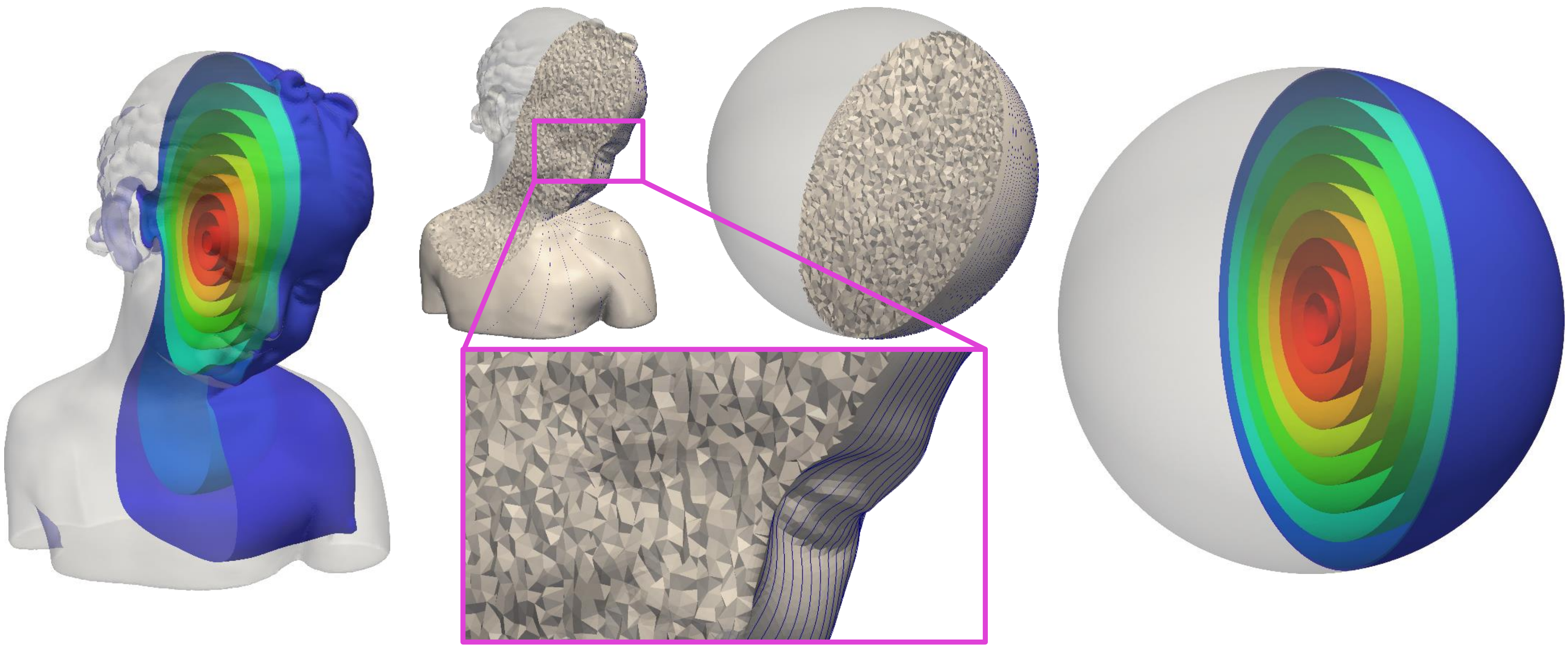
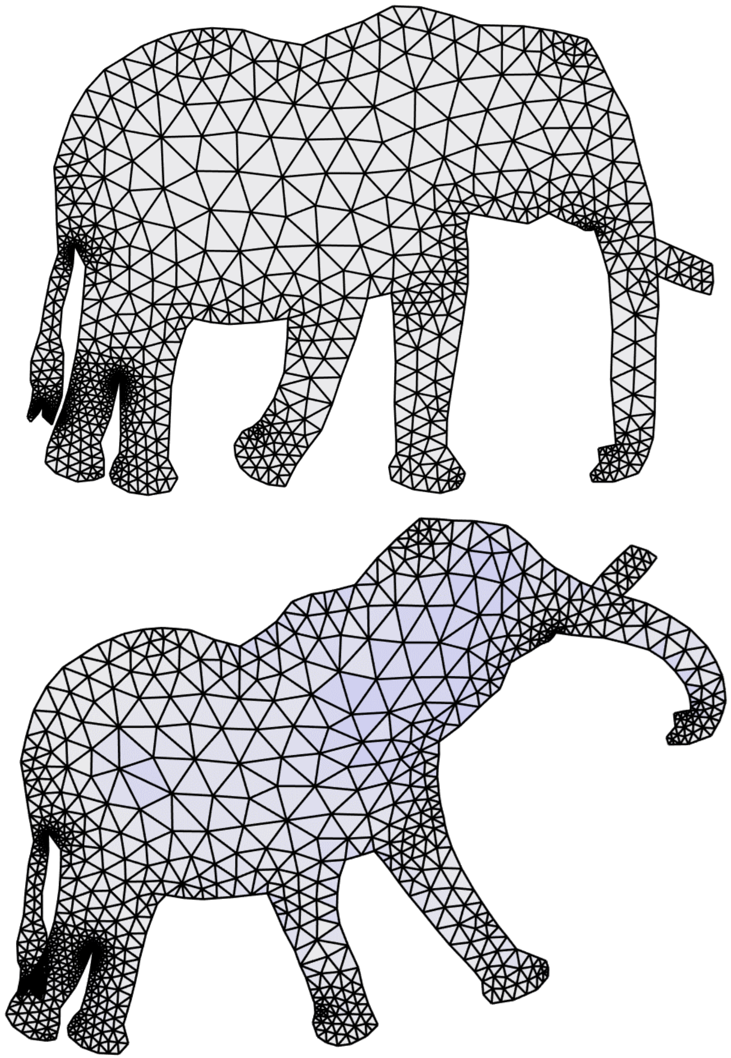
Large Scale Bounded Distortion Mappings

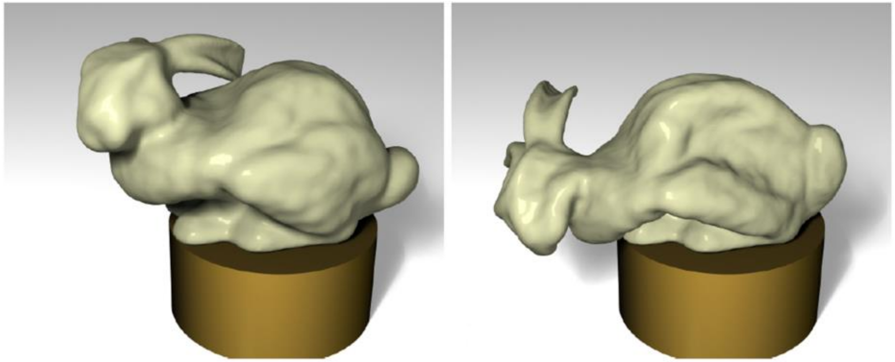
Shahar Kovalsky, Noam Aigerman, Ronen Basri and Yaron Lipman



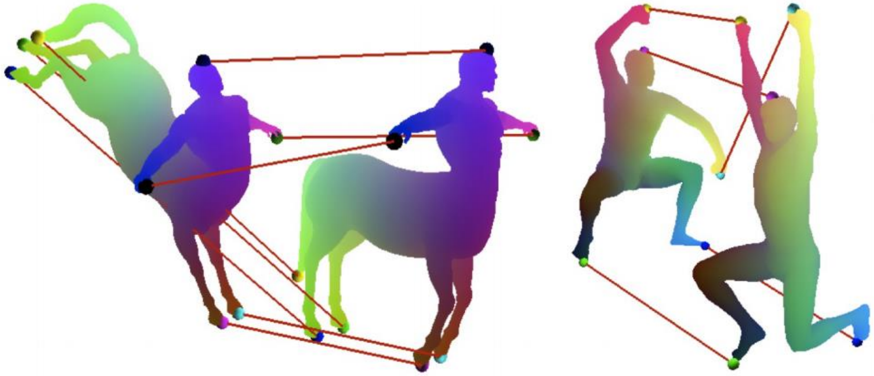
Mappings



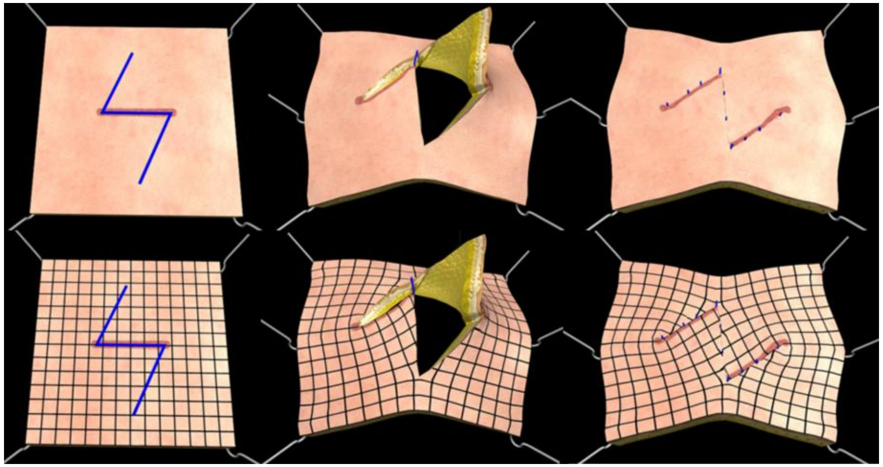
Applications



[Wang et al. 2010]

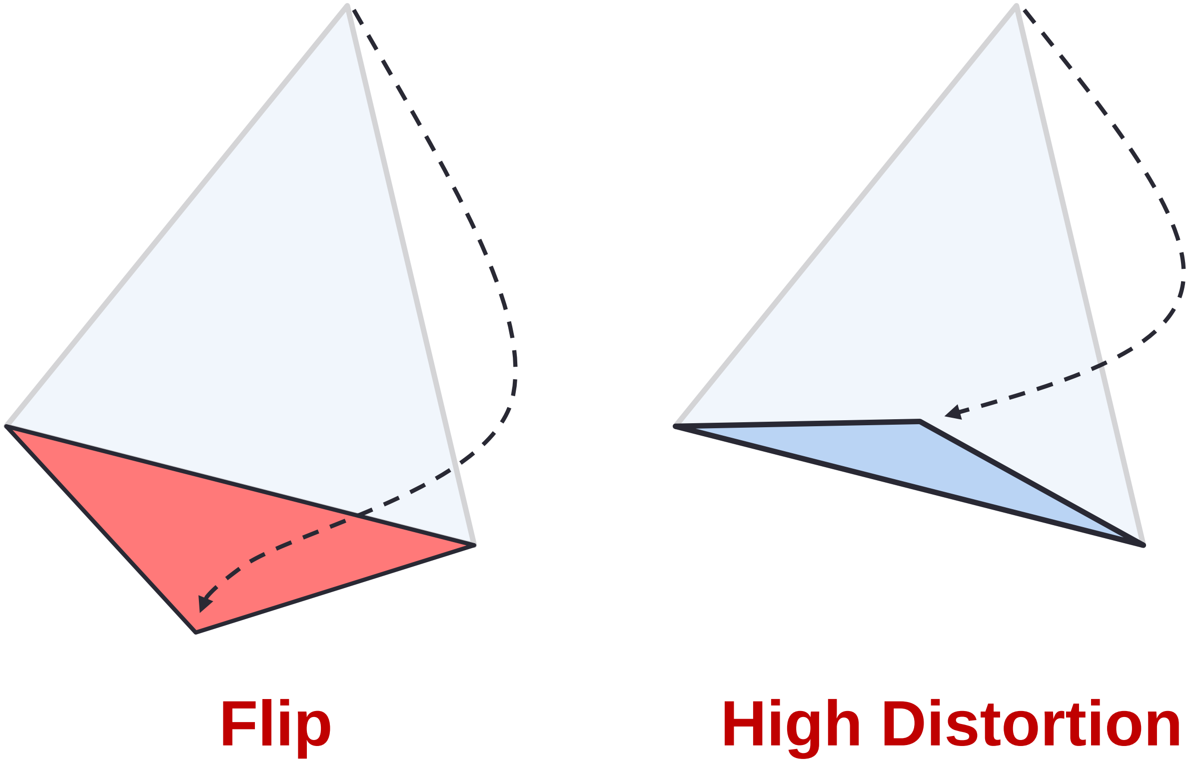


[Kim et al. 2010]

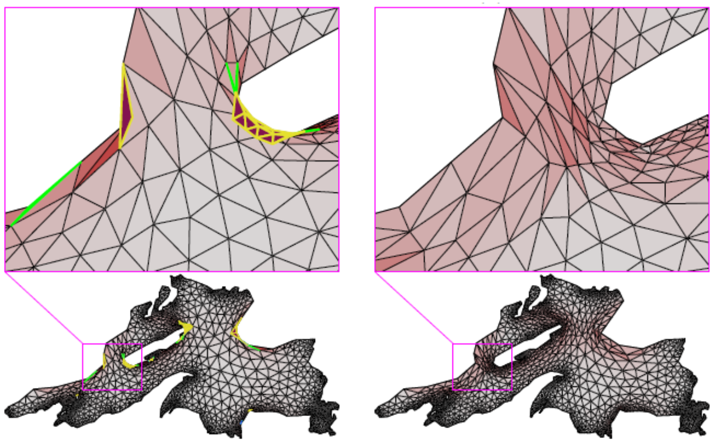


[Sifakis et al. 2009]

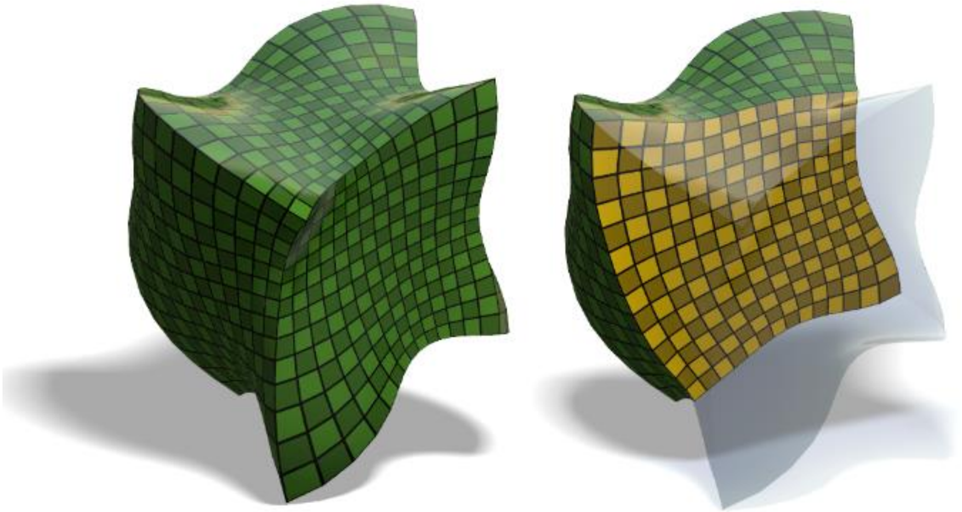
Misbehaved Mappings



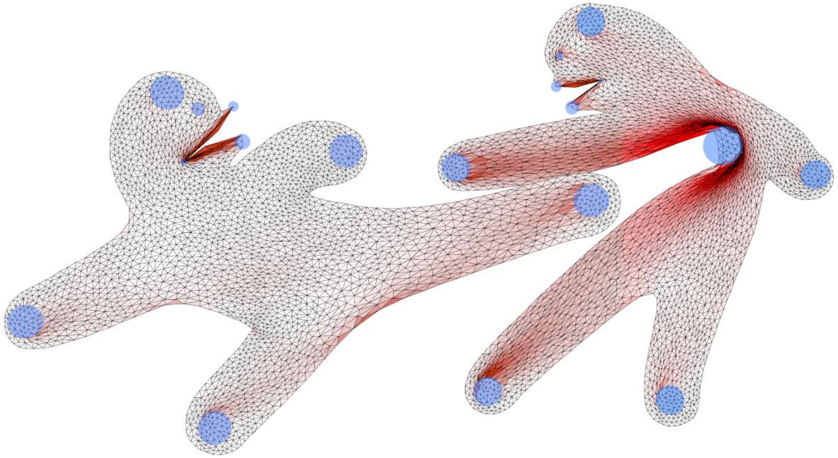
Related Work



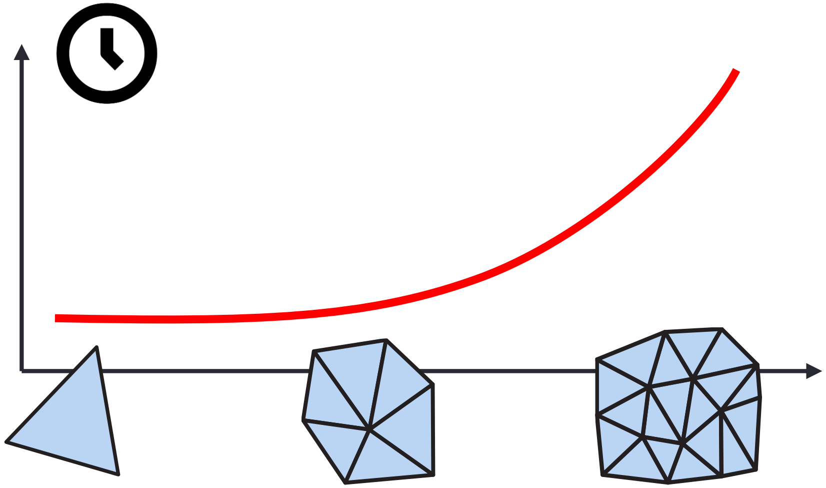
[Lipman 2012]



[Kovalsky et al. 2014]

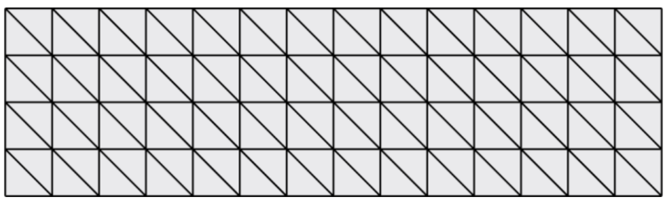


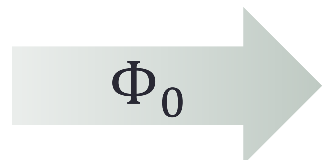
[Schüller et al. 2013]

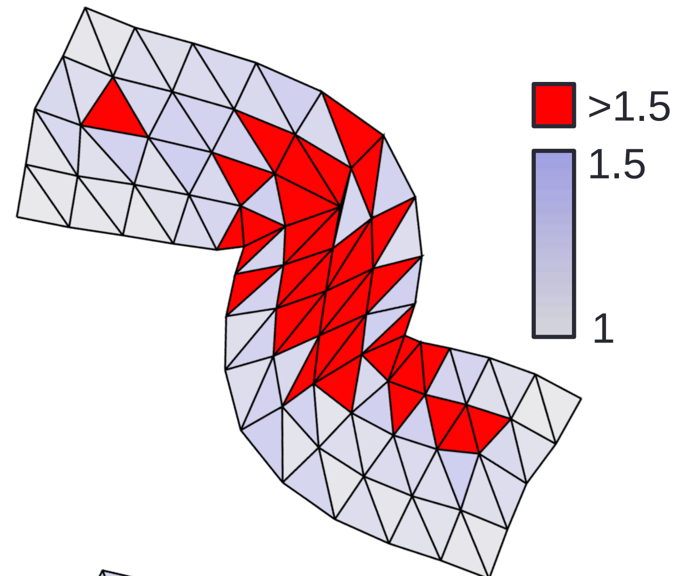


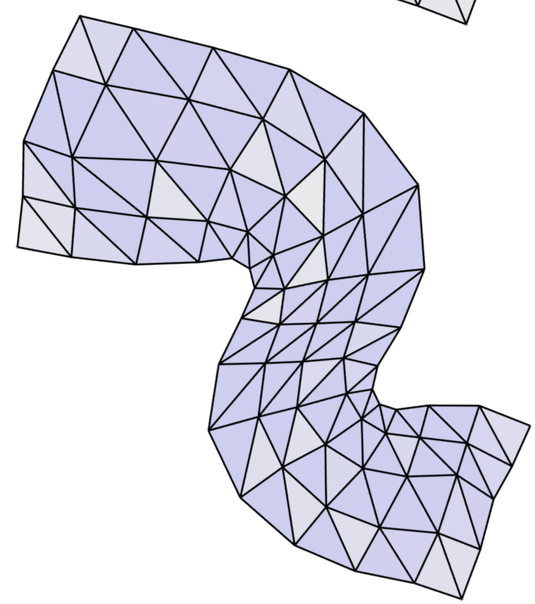
Goal

- Given:



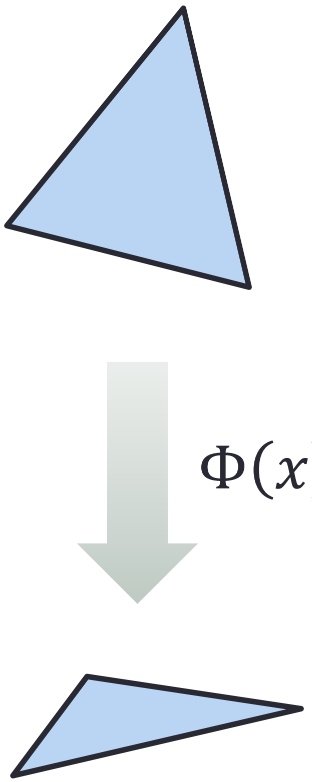




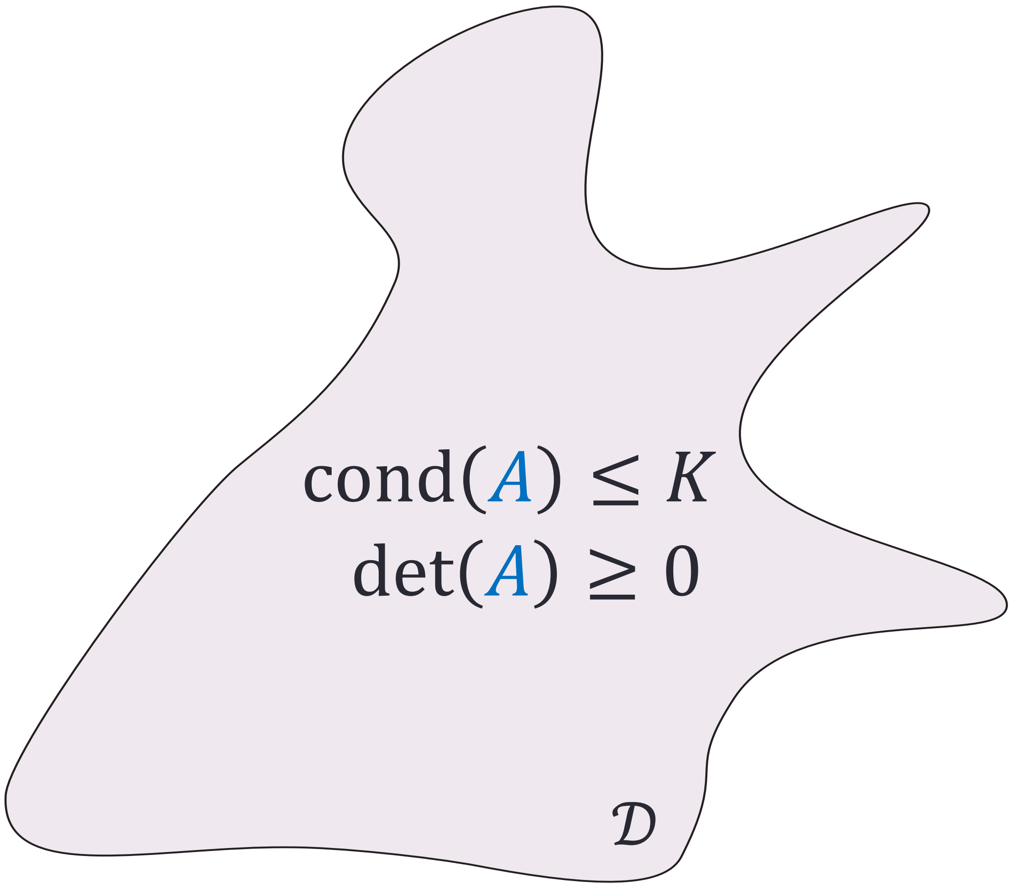


- Find $\Phi \approx \Phi_0$:
  - **No flips or high distortions**
  - **Fast and scalable**

Bounded Distortion

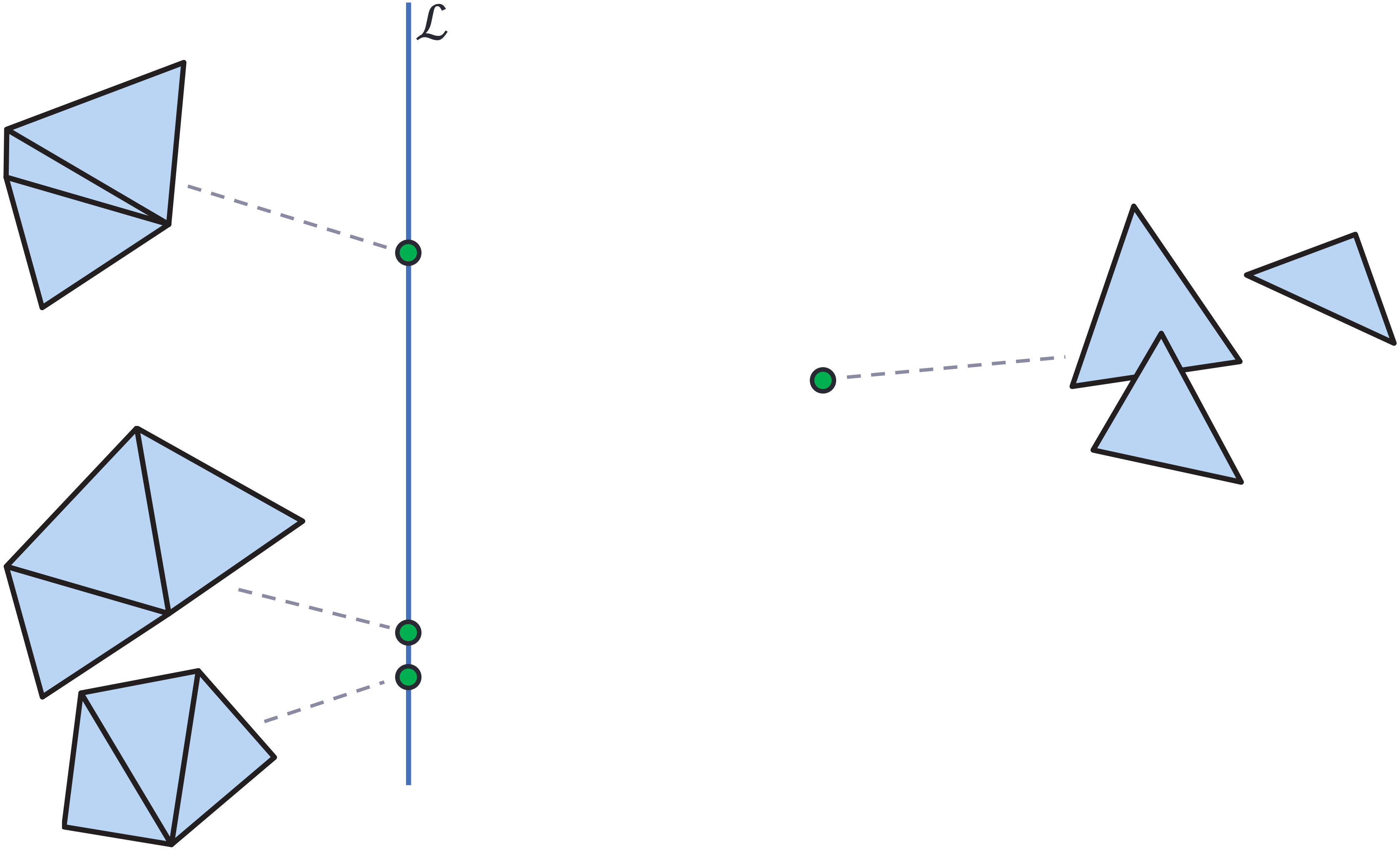


$$\Phi(x) = Ax + t$$

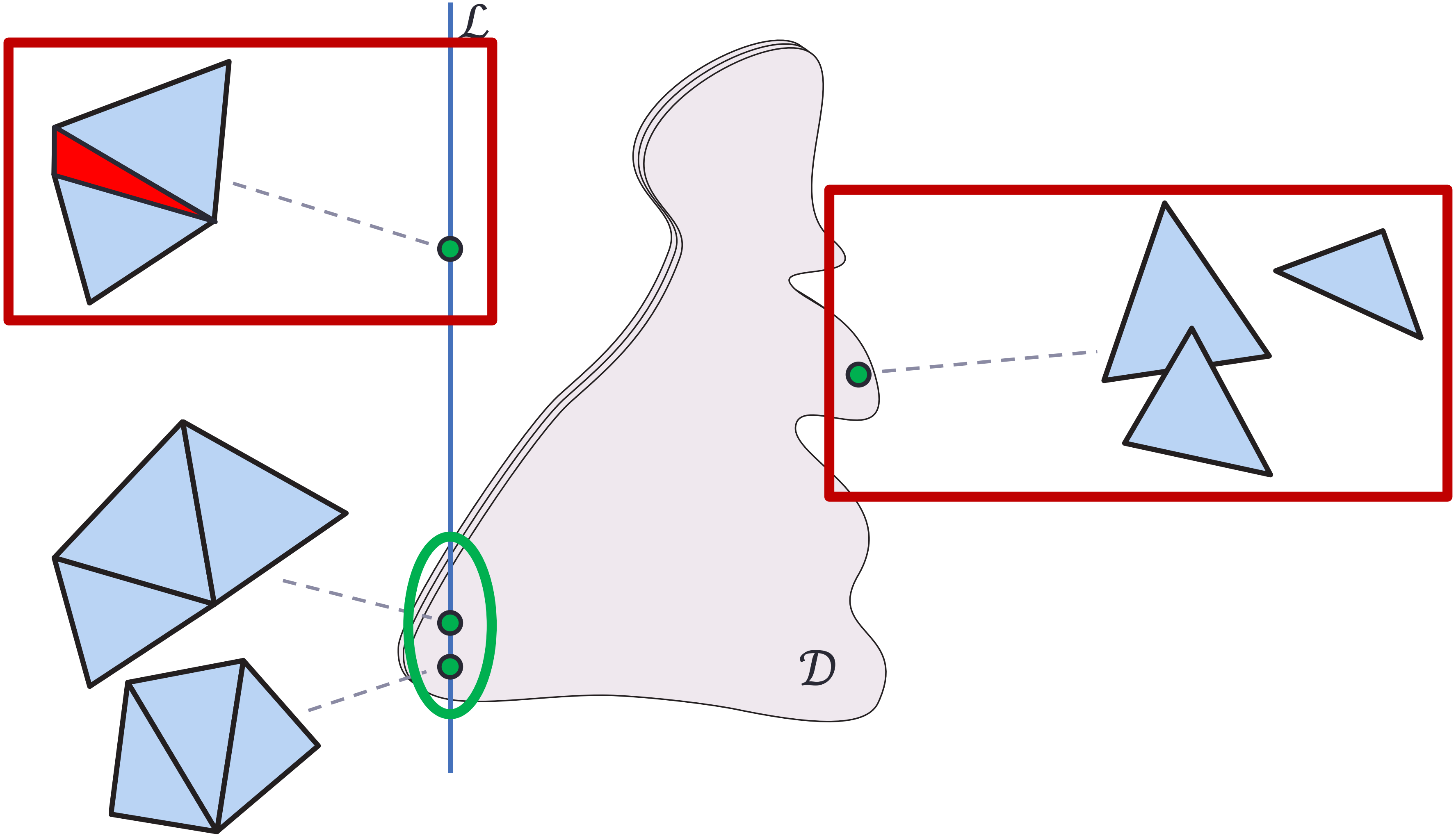


Scale-invariant characterization for
well-behaved mappings

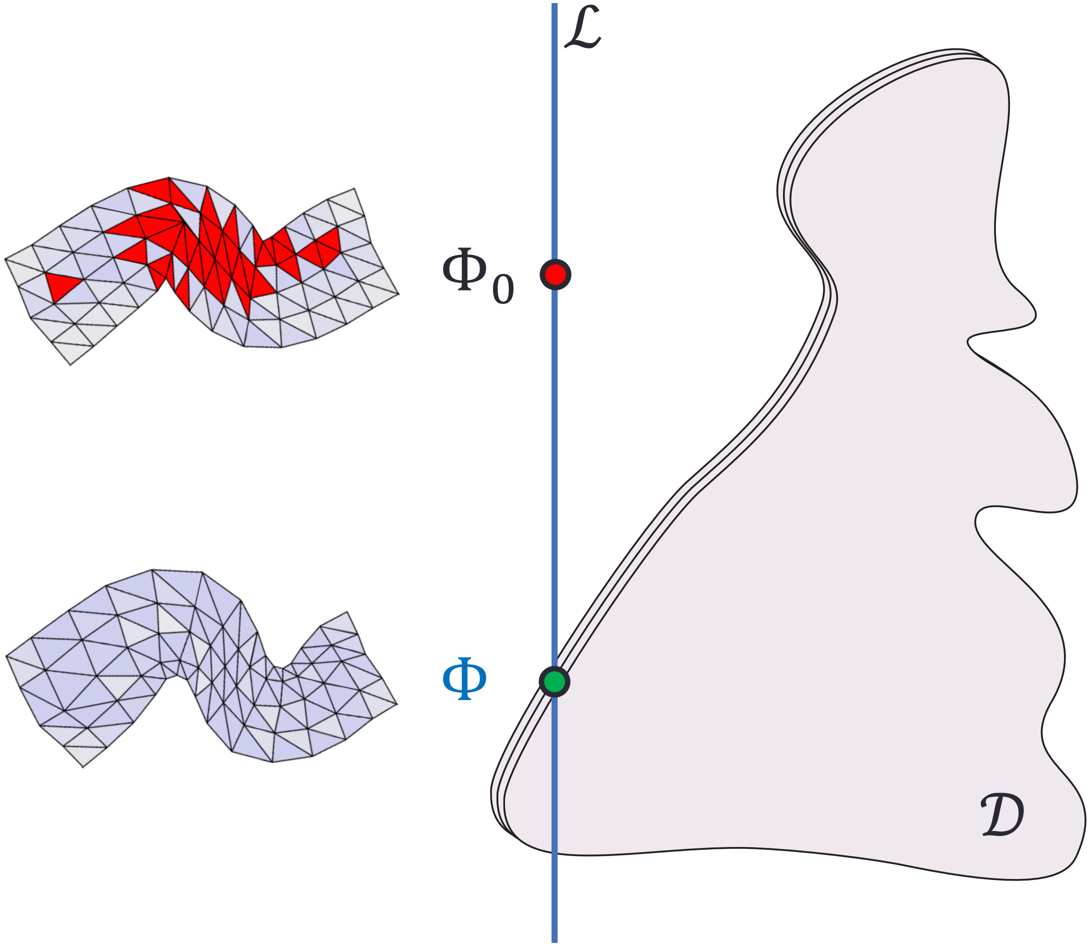
Bounded Distortion Simplicial Mappings



Bounded Distortion Simplicial Mappings



Goal

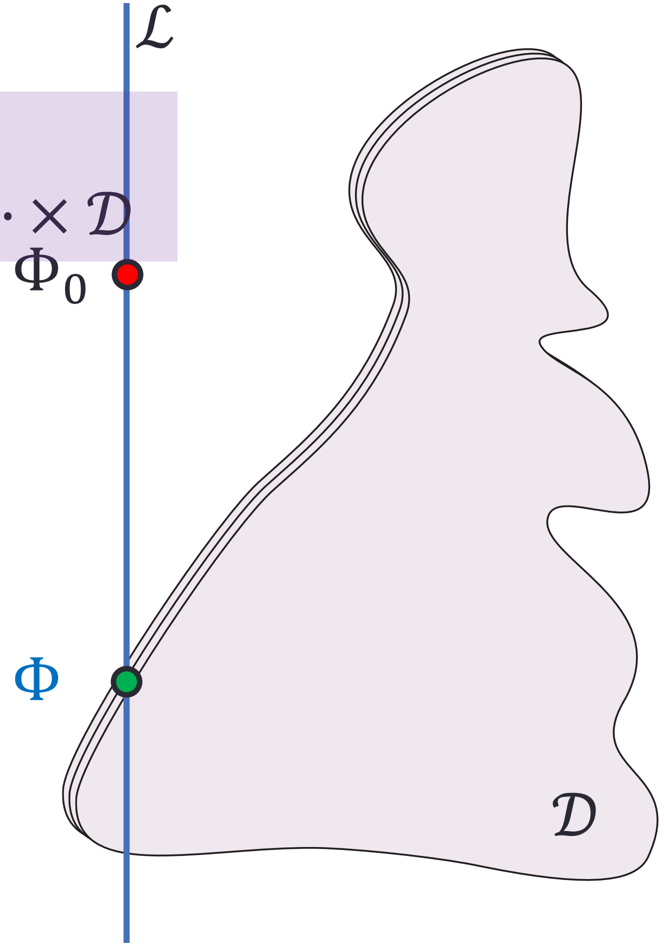


Goal

$$\min_{\Phi} \; \|\Phi - \Phi_0\|$$

s.t. $\Phi \in \mathcal{L}$

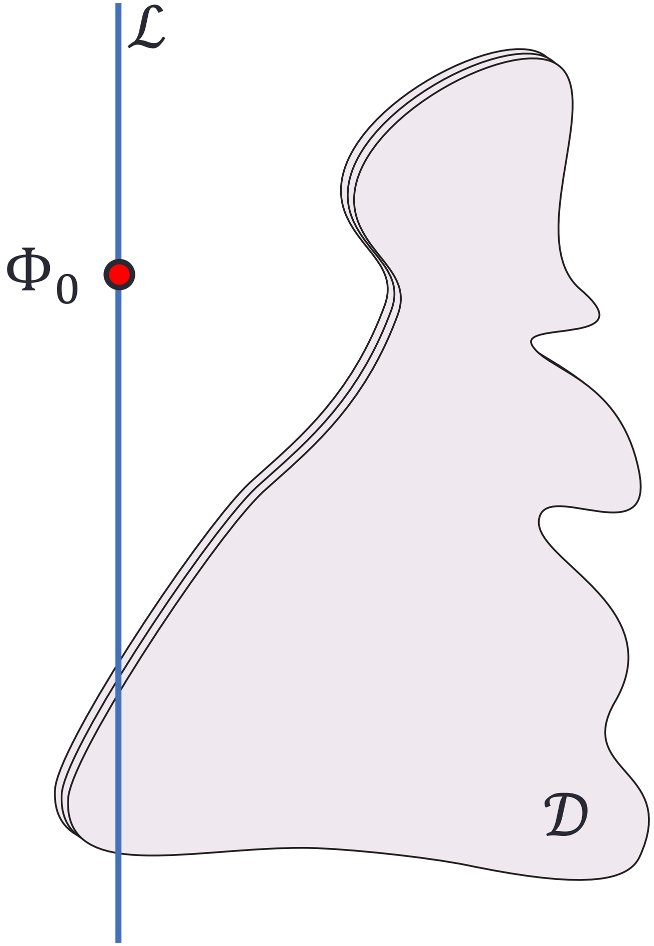
$\Phi \in \mathcal{D} \times \cdots \times \mathcal{D}$



The Challenge

$$\min_{\Phi} \ \|\Phi - \Phi_0\|$$
$$\text{s.t.} \quad \Phi \in \mathcal{L}$$
$$\Phi \in \mathcal{D} \times \cdots \times \mathcal{D}$$

efficient proxy for
$\Phi \in \mathcal{D} \times \cdots \times \mathcal{D}$??

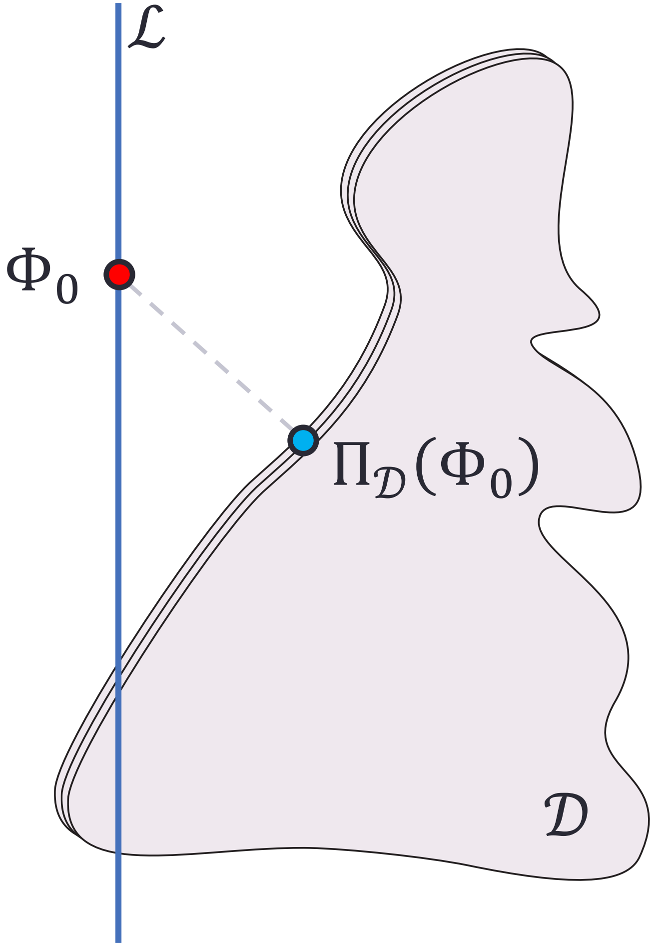


BD Projection

$$\min_{\Phi} \quad \|\Phi - \Phi_0\|$$

$$\text{s.t.} \quad \Phi \in \mathcal{L}$$

$$\Phi \in \mathcal{D} \times \cdots \times \mathcal{D}$$

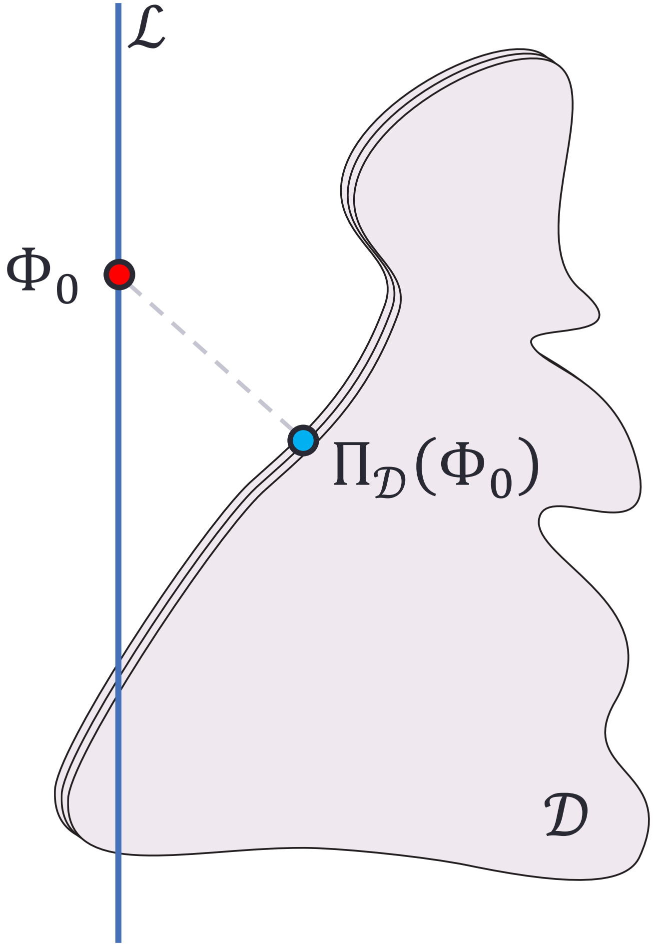


BD Projection

$$\min_{\Phi} \; \|\Phi - \Phi_0\|$$
$$\text{s.t.} \quad \Phi \in \mathcal{L}$$
$$\Phi \in \mathcal{D} \times \cdots \times \mathcal{D}$$

Projection onto BD

- Separable ⇒ Parallelizable
- Low dimensional SVD
  + simple arithmetic

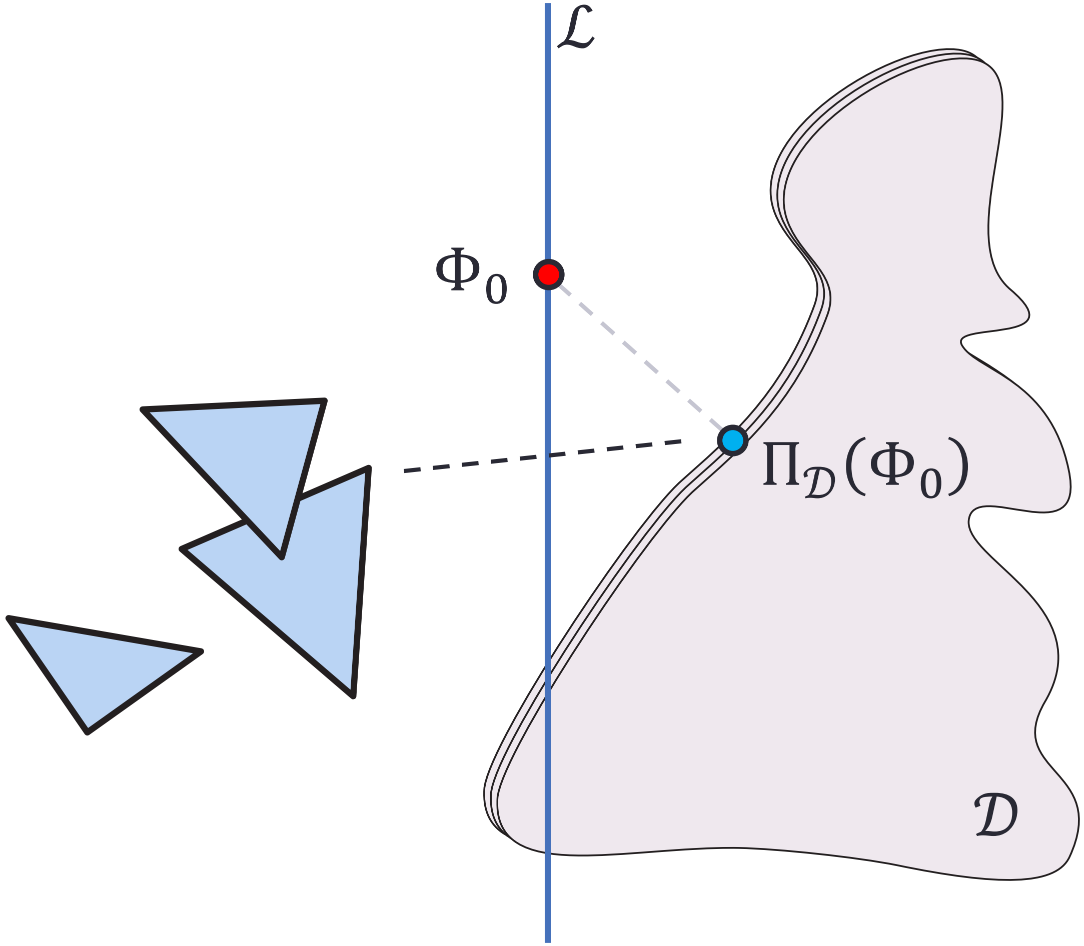


BD Projection

$$\min_{\Phi} \quad \|\Phi - \Phi_0\|$$

$$\text{s.t.} \quad \Phi \in \mathcal{L}$$

$$\Phi \in \mathcal{D} \times \cdots \times \mathcal{D}$$

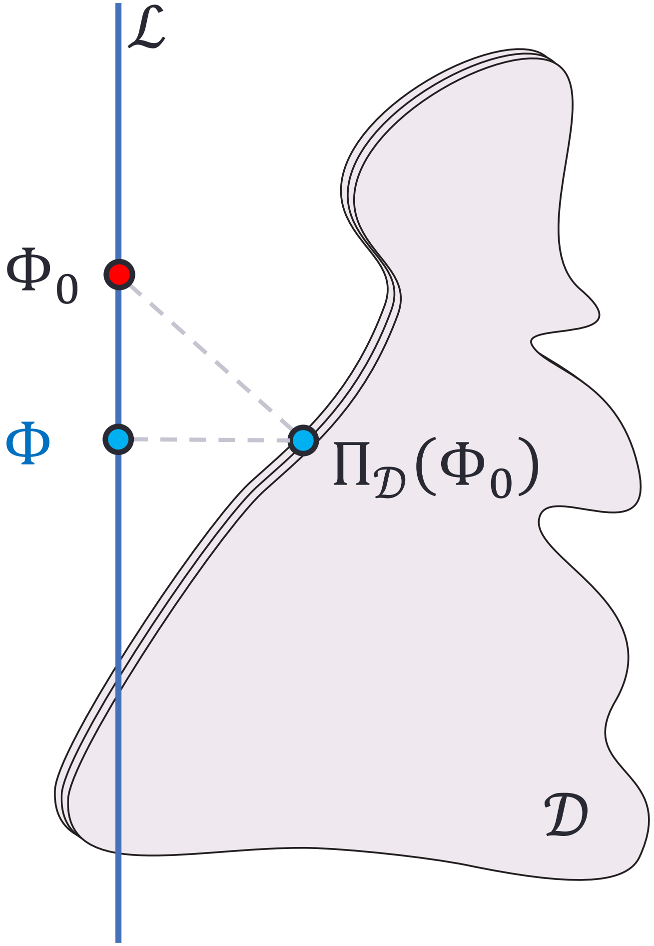


BD Projection

$$\min_{\Phi} \; \|\Phi - \Phi_0\|$$

$$\text{s.t.} \quad \Phi \in \mathcal{L}$$

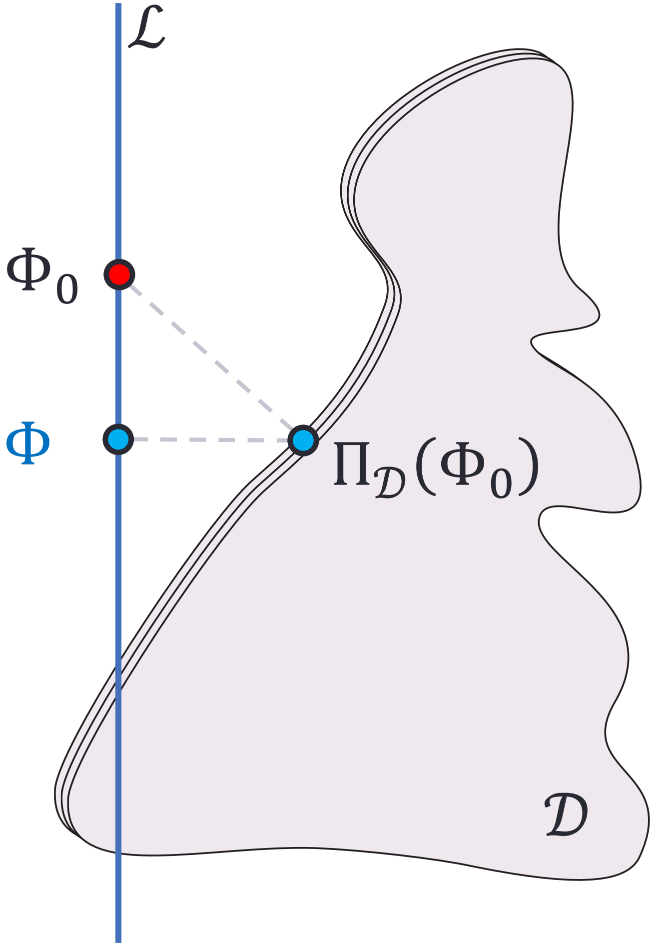
$$\Phi \in \mathcal{D} \times \cdots \times \mathcal{D}$$



BD Projection

$$\min_{\Phi} \quad \|\Phi - \Pi_{\mathcal{D}}(\Phi_0)\|$$

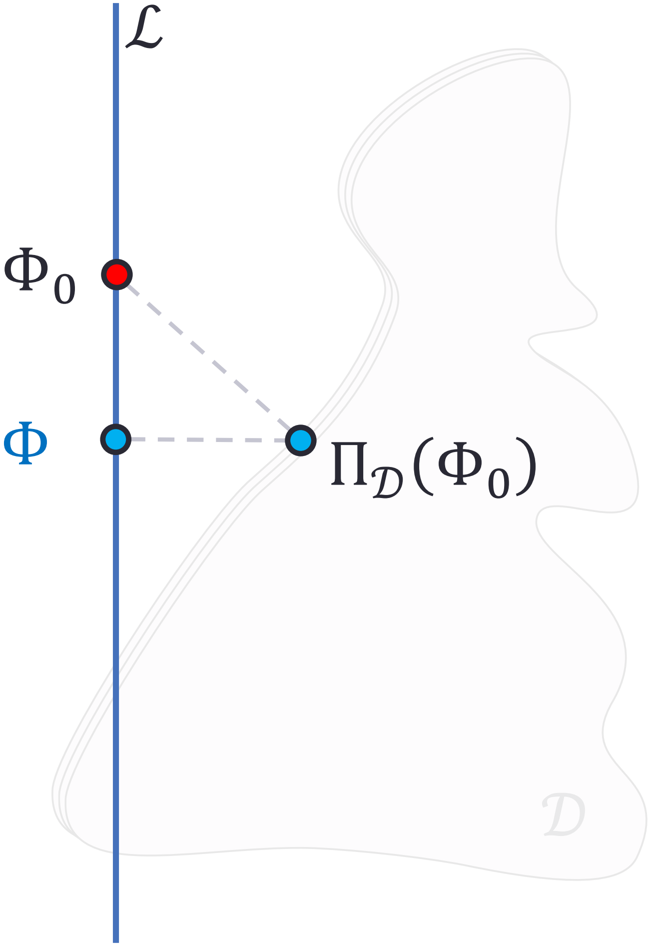
$$\text{s.t.} \quad \Phi \in \mathcal{L}$$

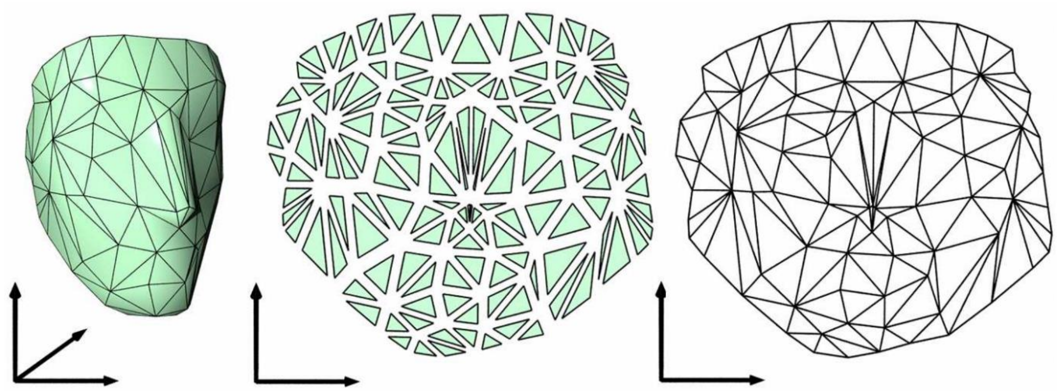


Alternating Optimization

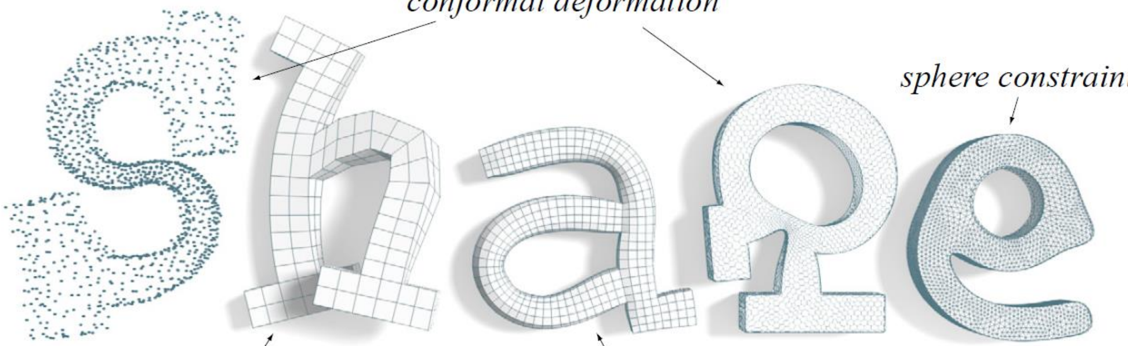
$$\min_{\Phi} \ \|\Phi - \Pi_{\mathcal{D}}(\Phi_0)\|$$
$$\text{s.t.} \quad \Phi \in \mathcal{L}$$

Point proxy
for
$\Phi \in \mathcal{D} \times \cdots \times \mathcal{D}$





[Liu et al. 2008]



[Bouaziz et al. 2012]

Alternating Optimization

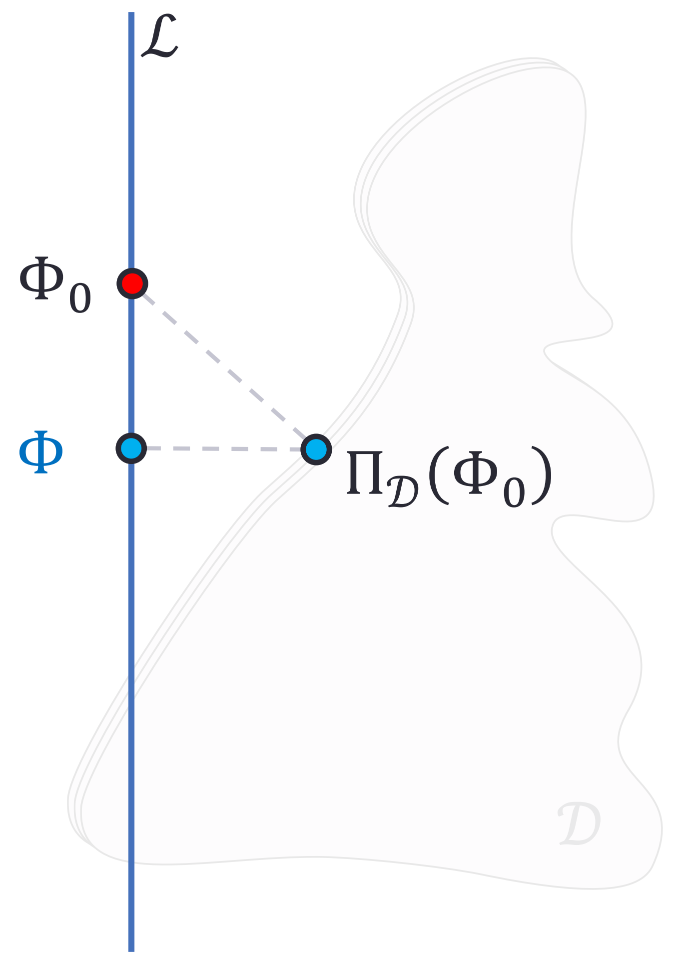
$$\min_{\Phi} \; \|\Phi - \Pi_{\mathcal{D}}(\Phi_0)\|$$
$$\text{s.t.} \quad \Phi \in \mathcal{L}$$

Solve a linear system:
$Ax = b$

Fixed

- Can factorize A
  $\Rightarrow$ Super efficient iterations

- Very slow convergence…

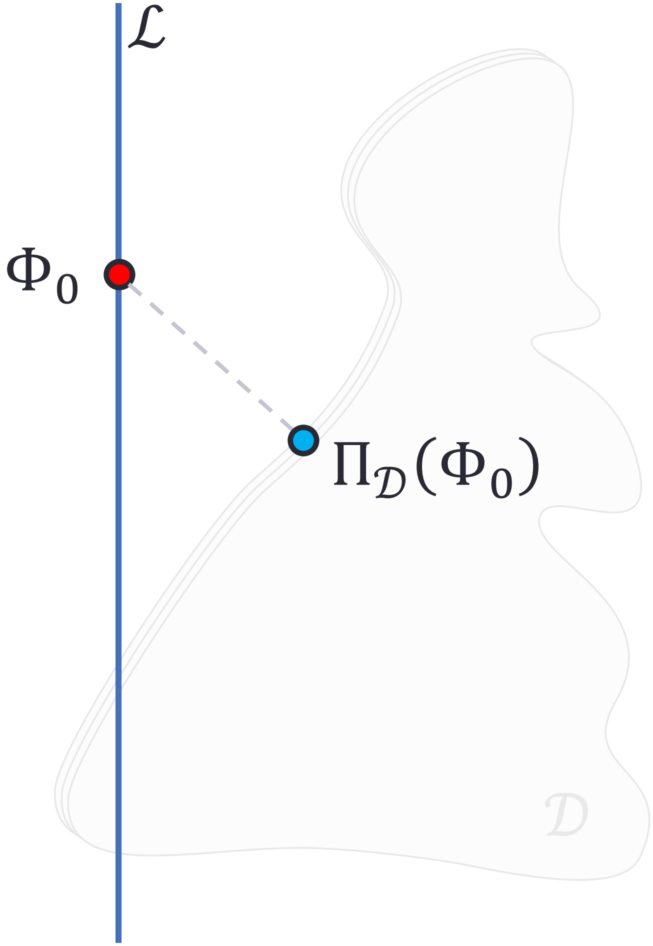


Alternating Optimization

$$\min_{\Phi} \quad \|\Phi - \Pi_{\mathcal{D}}(\Phi_0)\|$$

$$\text{s.t.} \quad \Phi \in \mathcal{L}$$

$$\Phi \in \mathcal{D} \times \cdots \times \mathcal{D}$$

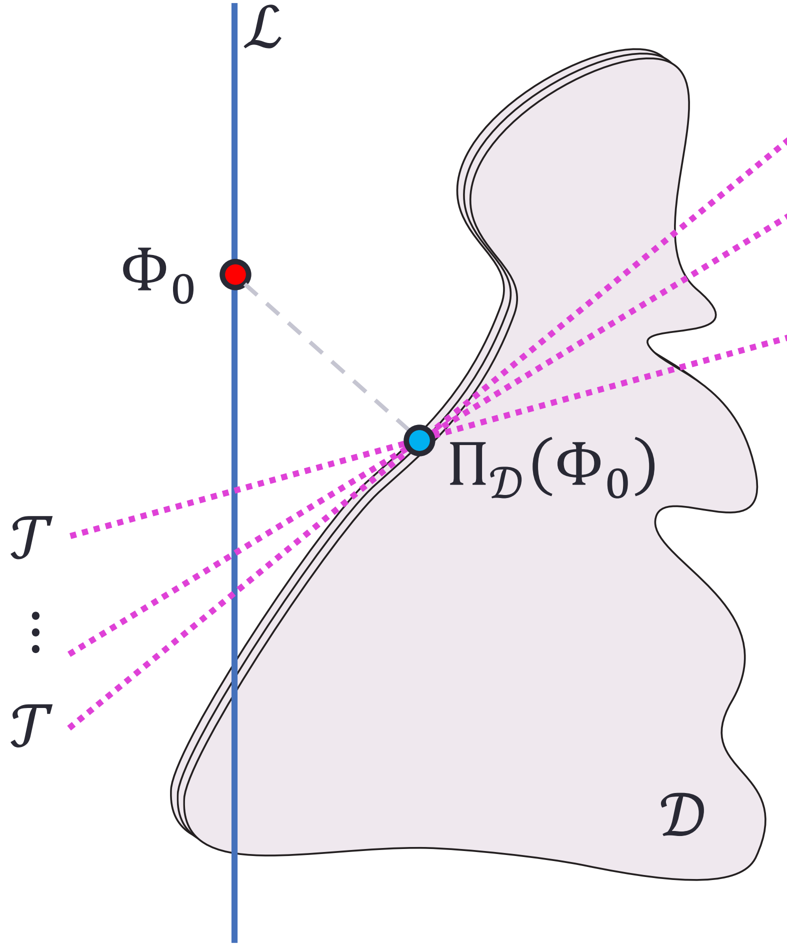


0th order vs. 1st order

$$\min_{\Phi} \quad \|\Phi - \Pi_{\mathcal{D}}(\Phi_0)\|$$

$$\text{s.t.} \quad \Phi \in \mathcal{L}$$

$$\Phi \in \mathcal{D} \times \cdots \times \mathcal{D}$$



Use Tangents

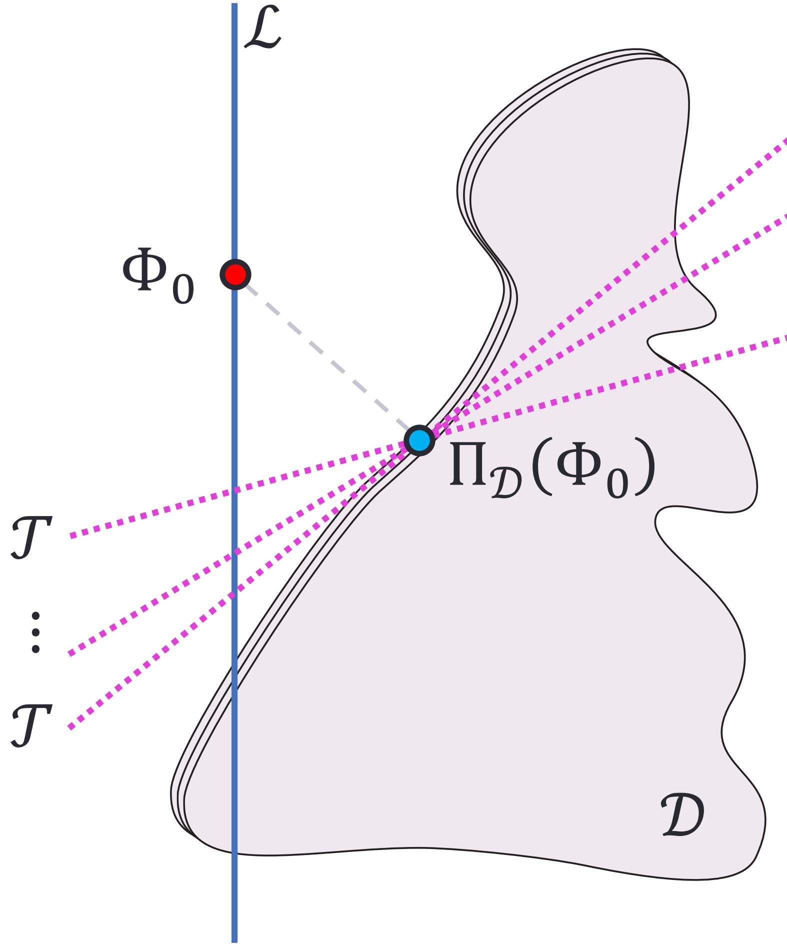
$$\min_{\Phi} \; \|\Phi - \Pi_{\mathcal{D}}(\Phi_0)\|$$

$$\text{s.t.} \quad \Phi \in \mathcal{L}$$

$$\Phi \in \mathcal{T} \times \cdots \times \mathcal{T}$$

1st order proxy for

$$\Phi \in \mathcal{D} \times \cdots \times \mathcal{D}$$



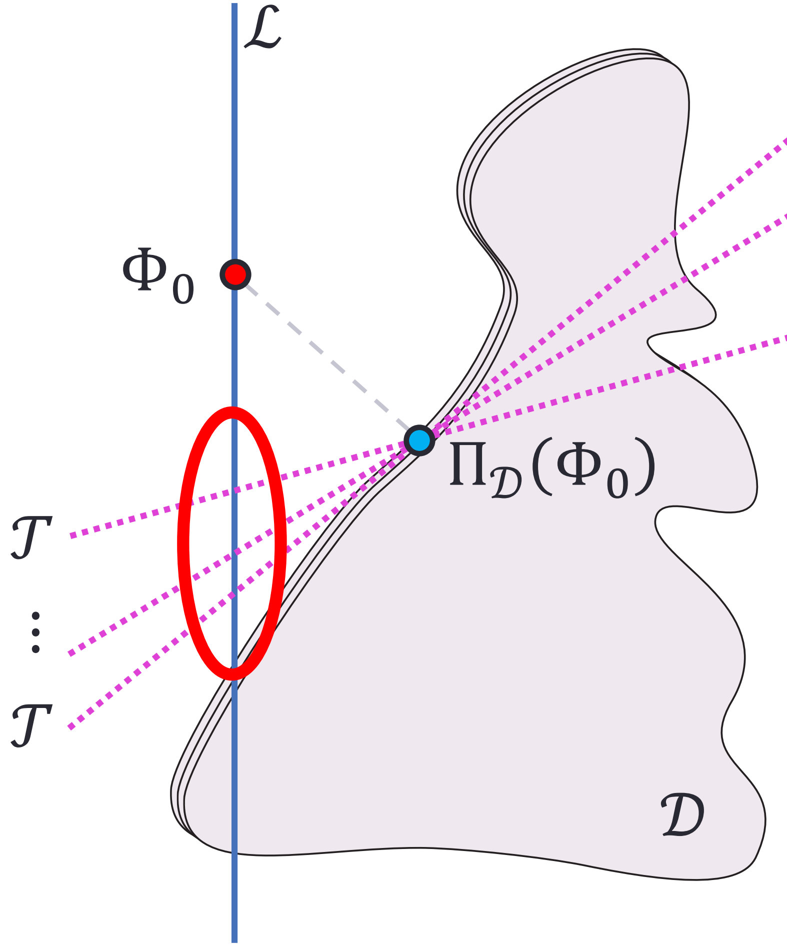
Use Tangents

$$\min_{\Phi} \; \|\Phi - \Pi_{\mathcal{D}}(\Phi_0)\|$$

s.t. $\Phi \in \mathcal{L}$

$\Phi \in \mathcal{T} \times \cdots \times \mathcal{T}$

Infeasible!

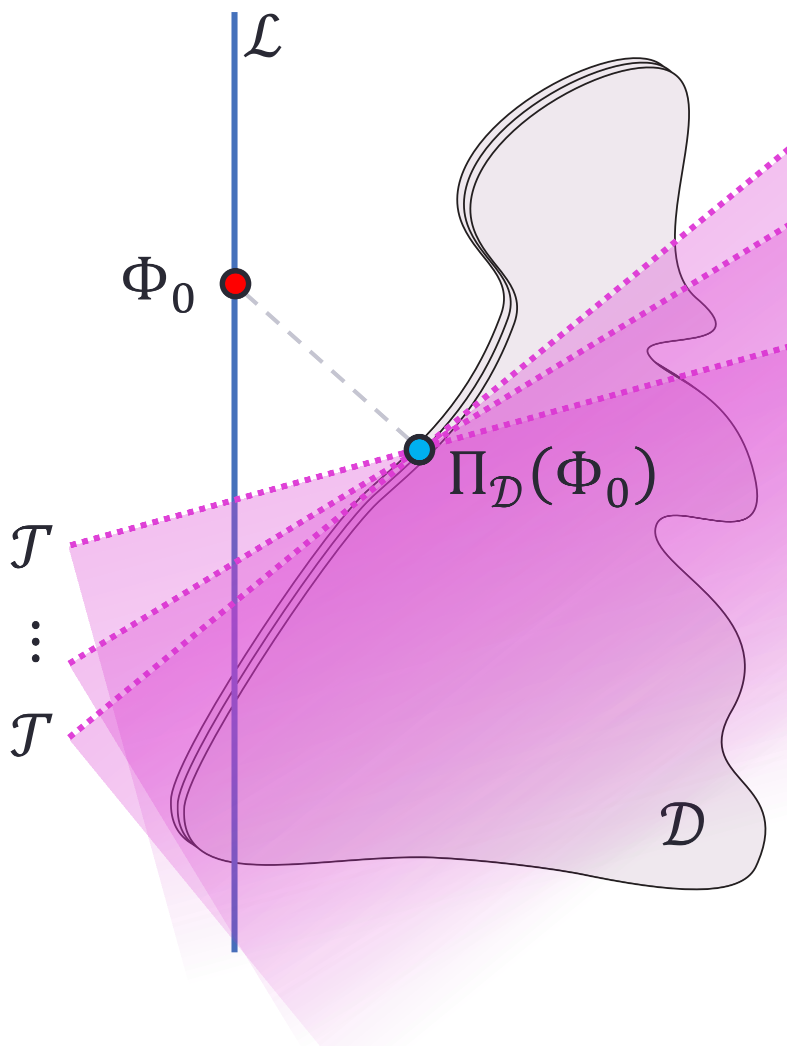


Use Tangents

$$\min_{\Phi} \ \|\Phi - \Pi_{\mathcal{D}}(\Phi_0)\|$$

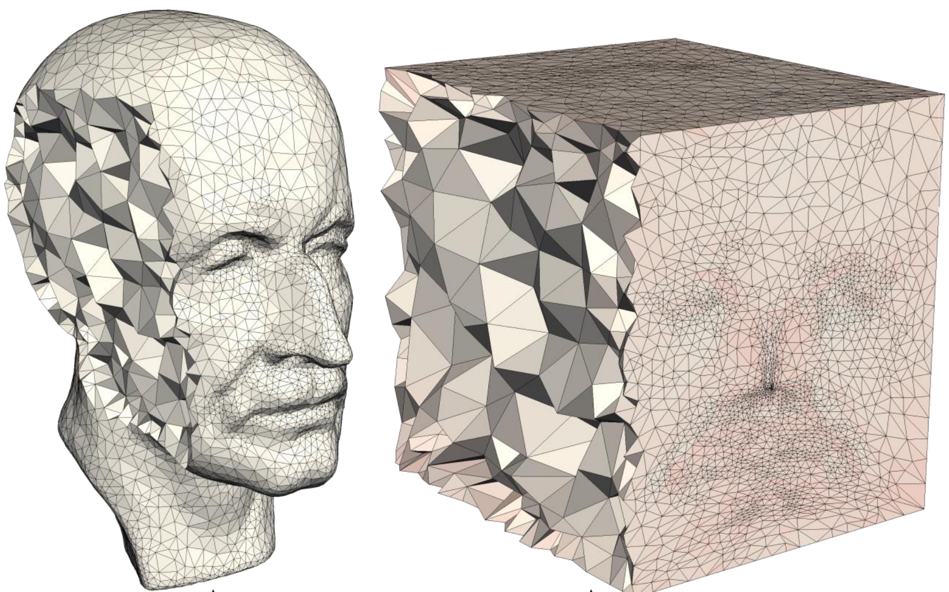
s.t. $\Phi \in \mathcal{L}$

$\Phi \in \mathcal{T} \times \cdots \times \mathcal{T}$

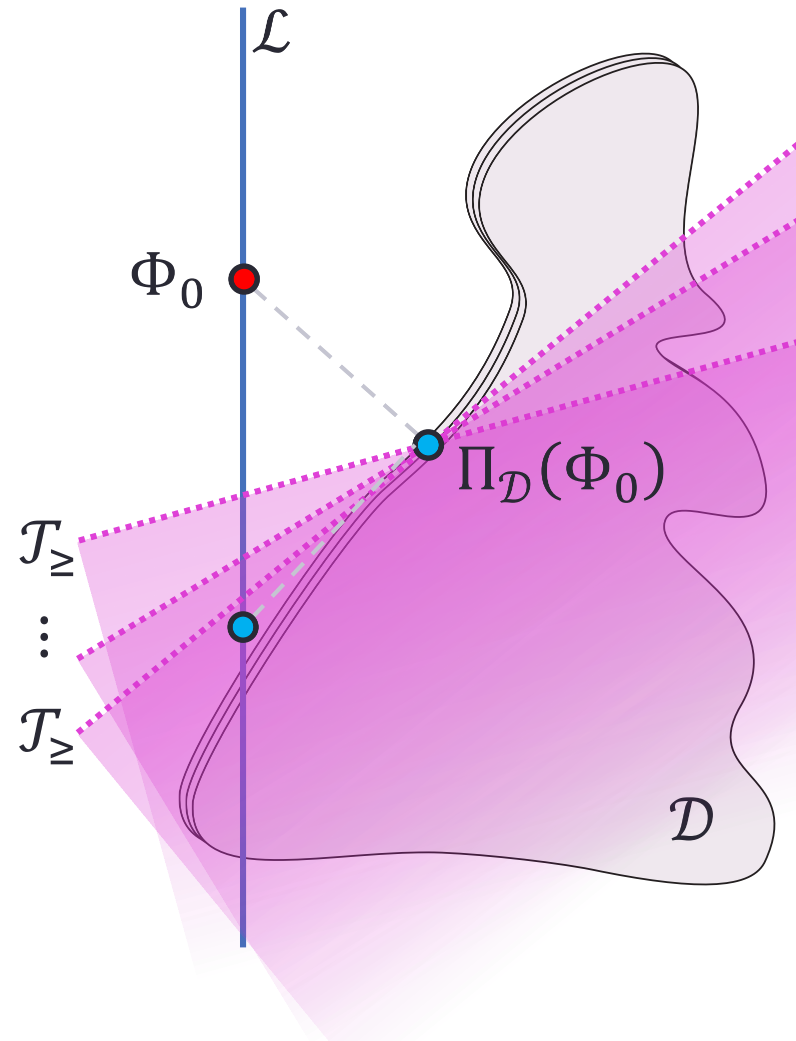


Use Tangents

$$\min_{\Phi} \ \|\Phi - \Pi_{\mathcal{D}}(\Phi_0)\|$$
$$\text{s.t.} \quad \Phi \in \mathcal{L}$$
$$\Phi \in \mathcal{T}_{\geq} \times \cdots \times \mathcal{T}_{\geq}$$



[Aigerman and Lipman 2013]



Aigerman and Lipman 2013

$$\min_{\Phi} \; \|\Phi - \Pi_{\mathcal{D}}(\Phi_0)\|$$

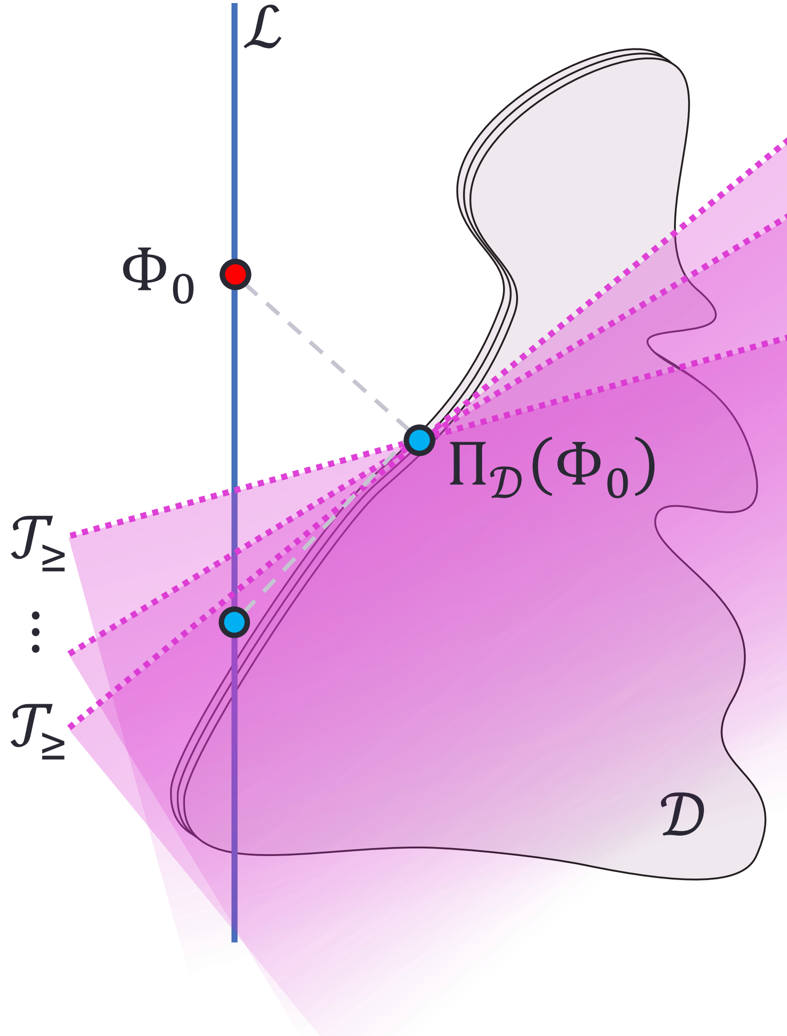
$$\text{s.t.} \quad \Phi \in \mathcal{L}$$

$$\Phi \in \mathcal{T}_{\geq} \times \cdots \times \mathcal{T}_{\geq}$$

Solve a Quadratic Program

- Quadratic convergence

- Uses interior point solver
- Poor scalability



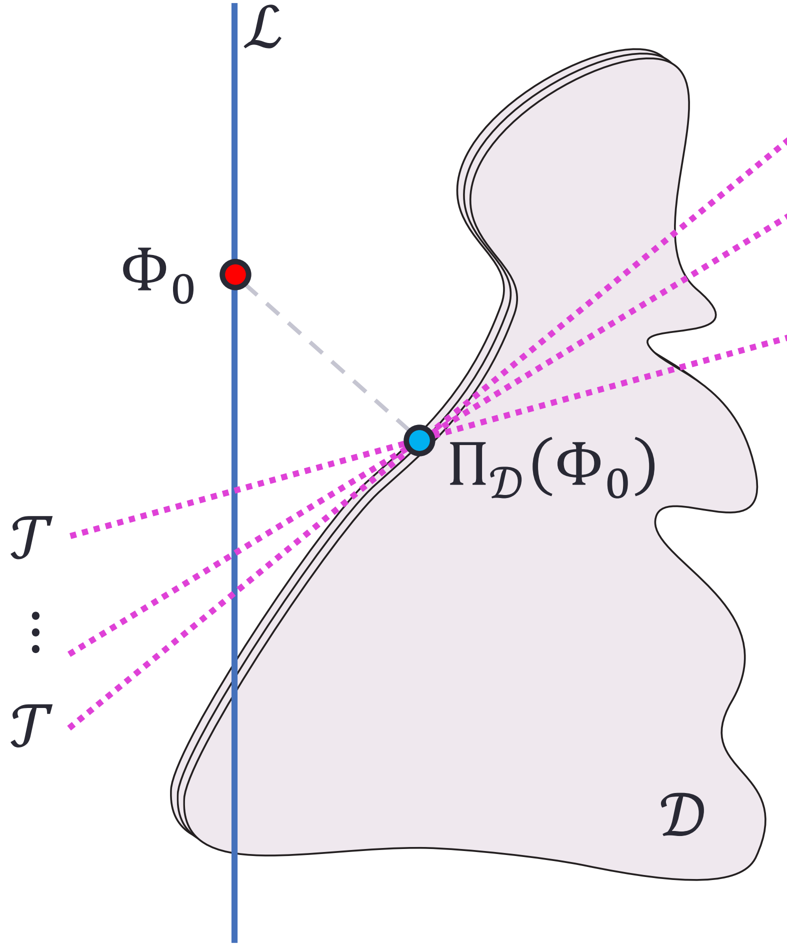
Use Tangents

$$\min_{\Phi} \ \|\Phi - \Pi_{\mathcal{D}}(\Phi_0)\|$$

s.t. $\Phi \in \mathcal{L}$

$\Phi \in \mathcal{T} \times \cdots \times \mathcal{T}$

Infeasible!

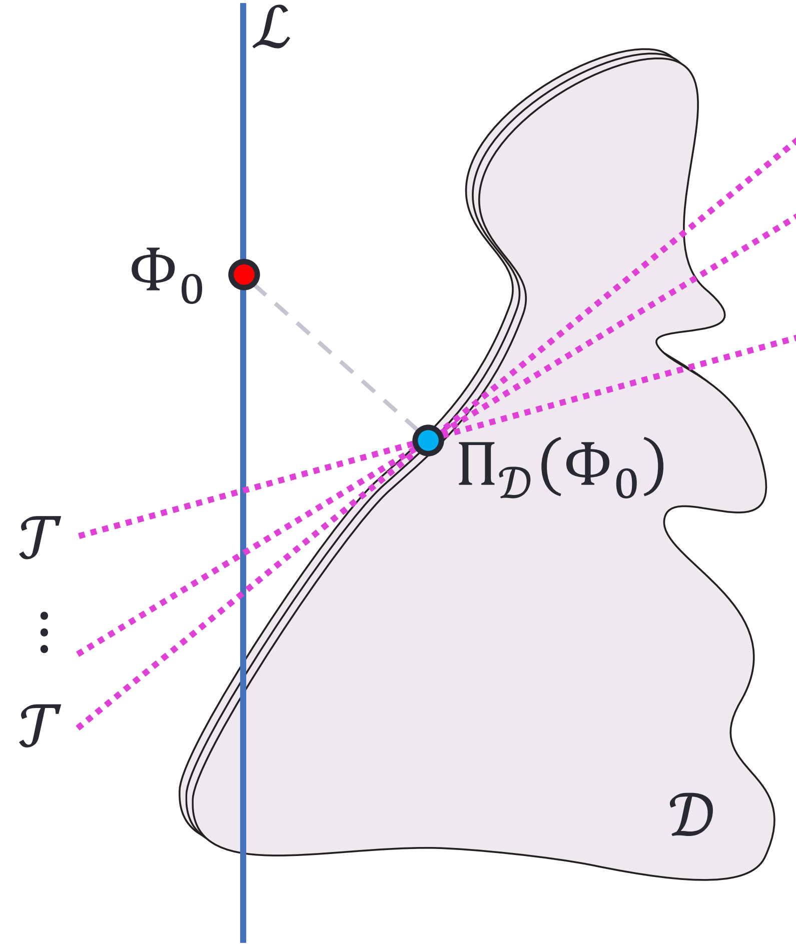


Non-linear Least Squares

$$\min_{\Phi} \; \|\Phi - \Pi_{\mathcal{D}}(\Phi_0)\|$$

$$\text{s.t.} \quad \Phi \in \mathcal{L}$$

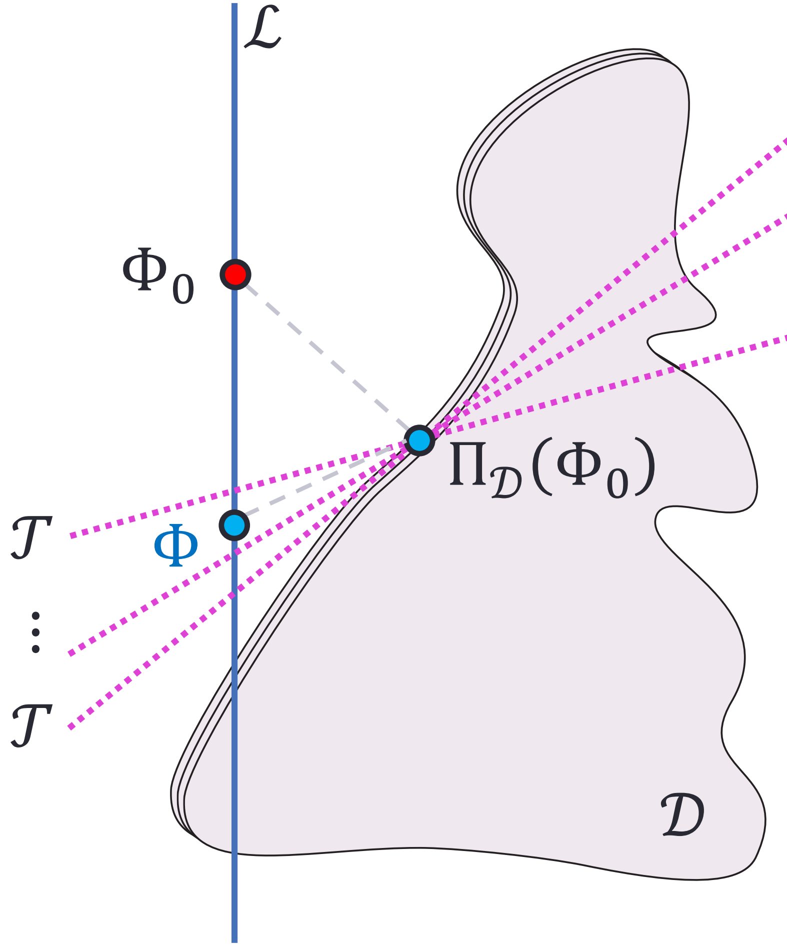
$$\Phi \in \mathcal{T} \times \cdots \times \mathcal{T}$$



Non-linear Least Squares

$$\min_{\Phi} \; \|\Phi - \Pi_{\mathcal{D}}(\Phi_0)\|^2 + \sum \lambda_j \left\langle \Phi_j, \mathcal{T}_j^{\perp} \right\rangle^2$$

$$\text{s.t.} \quad \Phi \in \mathcal{L}$$



Non-linear Least Squares

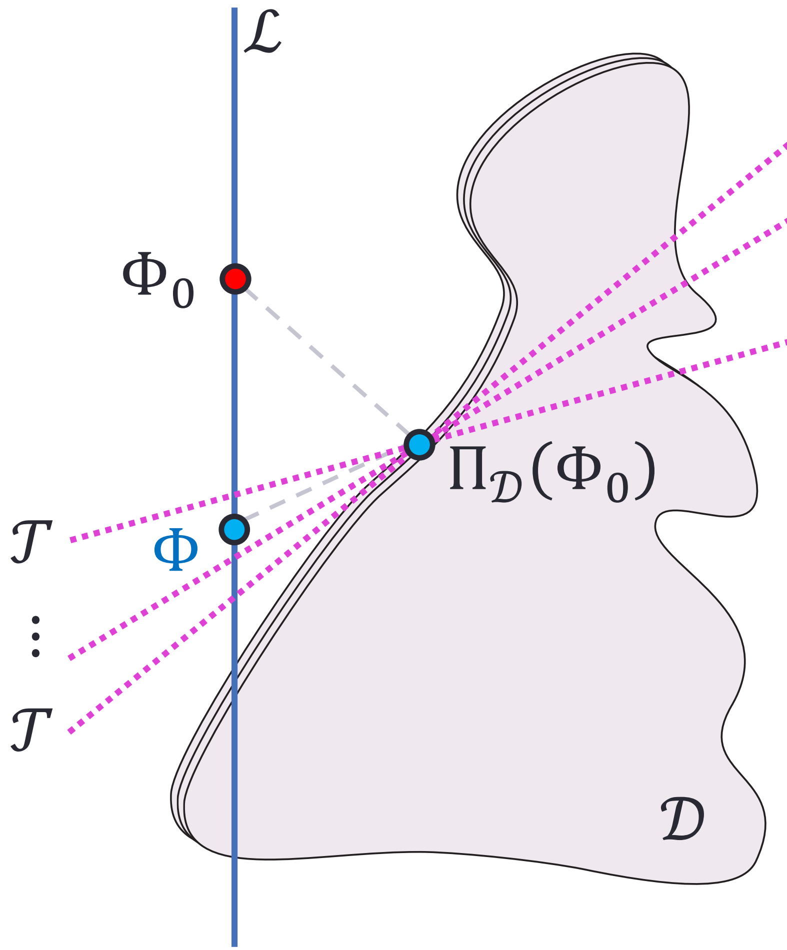
$$\min_{\Phi} \; \|\Phi - \Pi_{\mathcal{D}}(\Phi_0)\|^2 + \sum \lambda_j \langle \Phi_j, \mathcal{T}_j^{\perp} \rangle^2$$
$$\text{s.t.} \quad \Phi \in \mathcal{L}$$

Solve a linear system

(Gauss-Newton \ Levenberg-Marquardt)

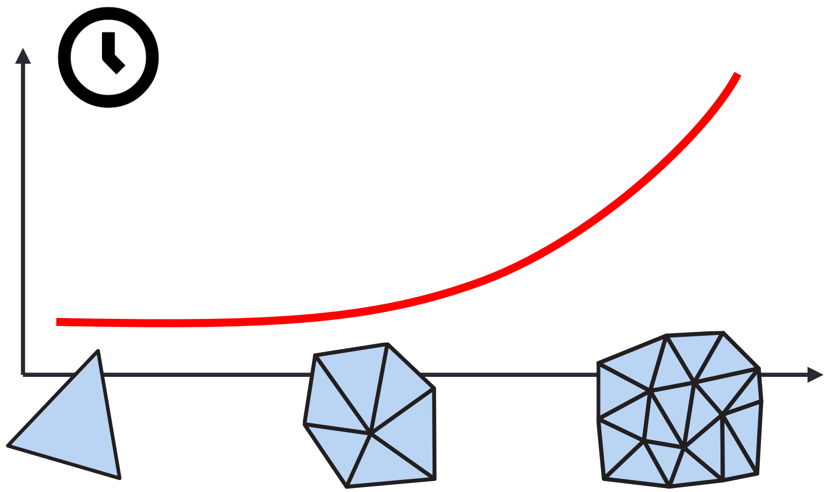
- 2nd-order-**like** convergence

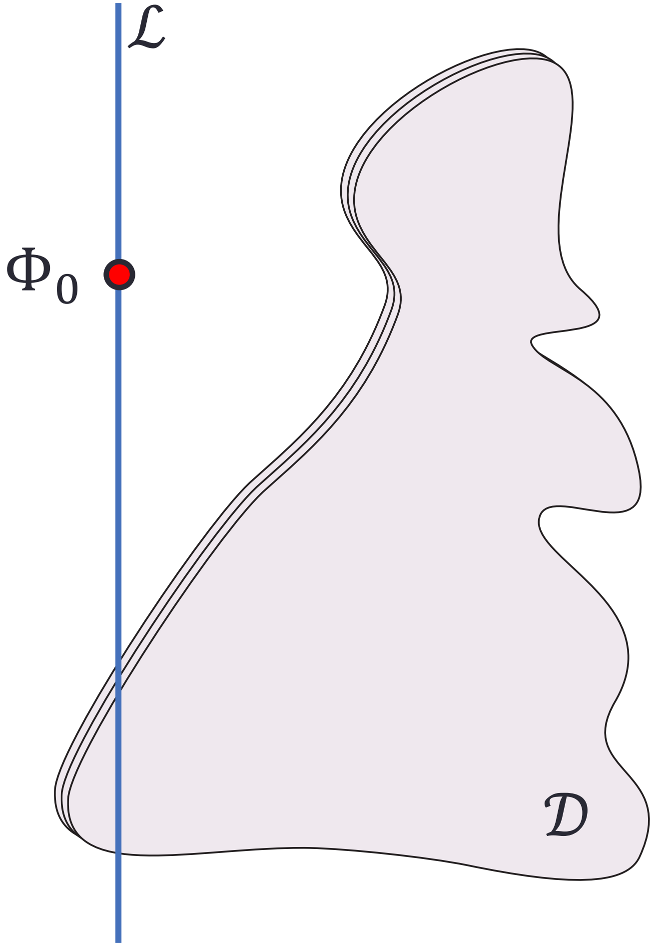
- Sensitive parameters
- Varying linear system



Our Approach

$$\min_{\Phi} \ \|\Phi - \Pi_{\mathcal{D}}(\Phi_0)\|$$
$$\text{s.t.} \quad \Phi \in \mathcal{L}$$
$$\Phi \in \mathcal{D} \times \cdots \times \mathcal{D}$$





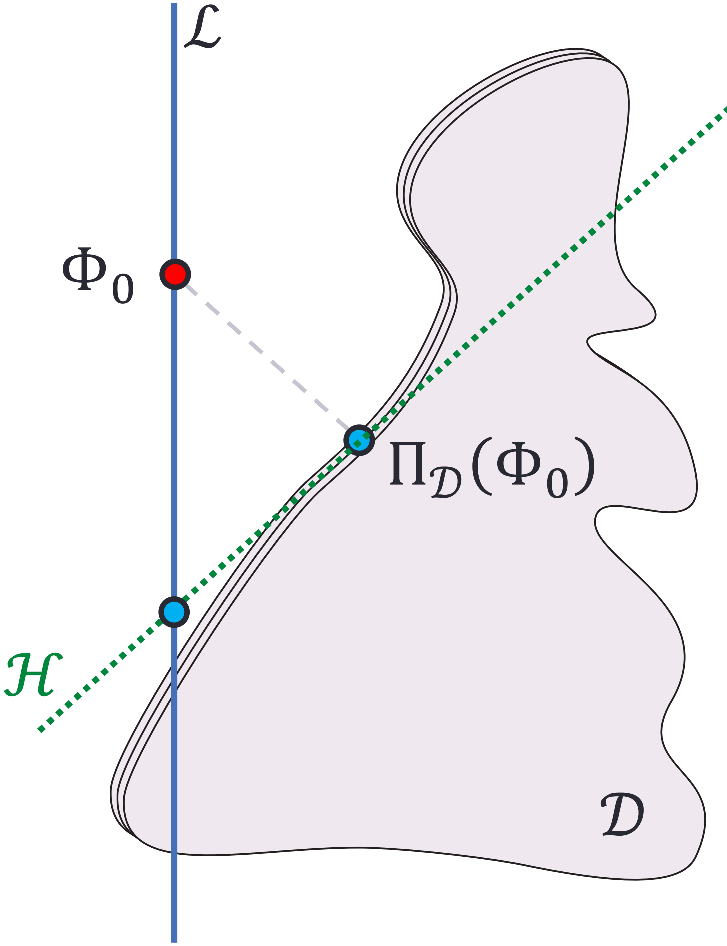
Our Approach

$$\min_{\Phi} \; \|\Phi - \Pi_{\mathcal{D}}(\Phi_0)\|$$

$$\text{s.t.} \quad \Phi \in \mathcal{L}$$

$$\Phi \in \underbrace{\mathcal{D} \times \cdots \times \mathcal{D}}_{\bar{\mathcal{D}}}$$

Think of a **single** high-dimensional set

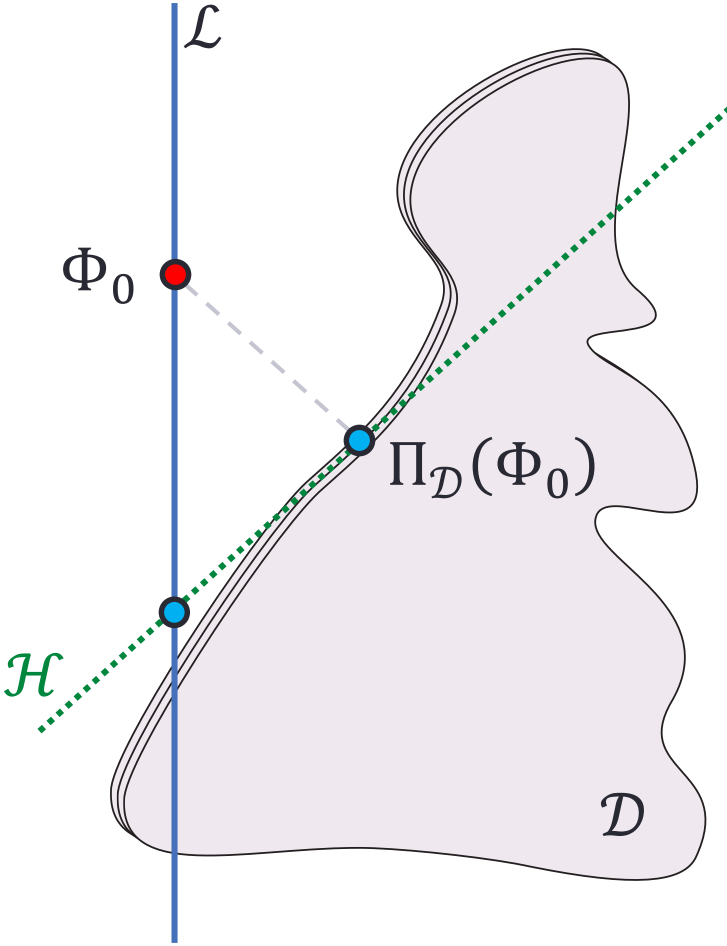


Our Approach

$$\min_{\Phi} \quad \|\Phi - \Pi_{\mathcal{D}}(\Phi_0)\|$$

$$\text{s.t.} \quad \Phi \in \mathcal{L}$$

$$\Phi \in \mathcal{D} \times \cdots \times \mathcal{D}$$



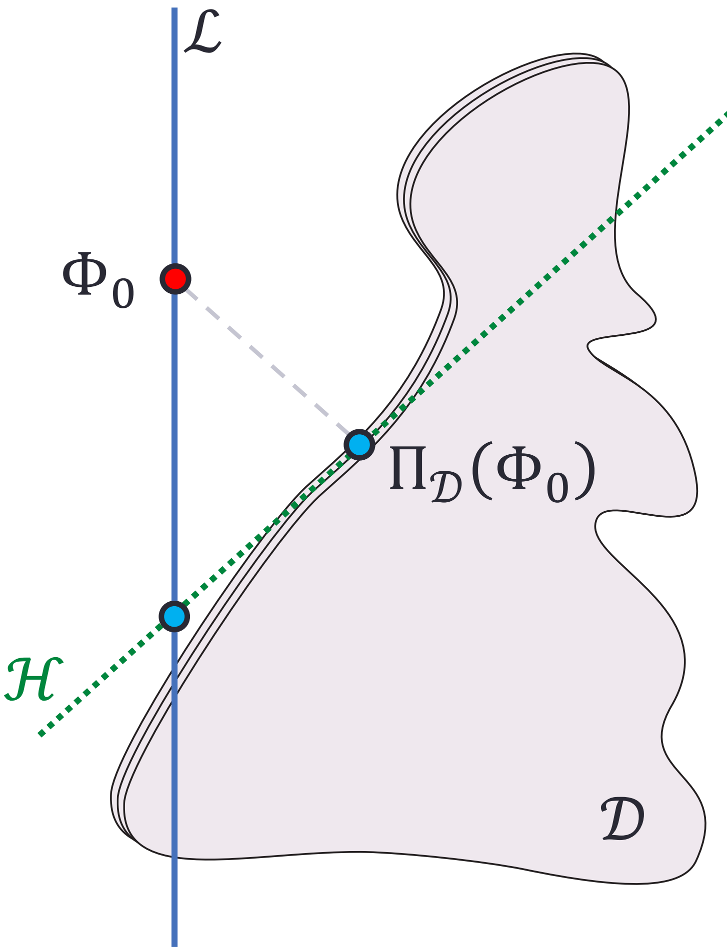
Our Approach

$$\min_{\Phi} \; \|\Phi - \Pi_{\mathcal{D}}(\Phi_0)\|$$

$$\text{s.t.} \quad \Phi \in \mathcal{L}$$

$$\Phi \in \mathcal{H}$$

1st-order **proxy**

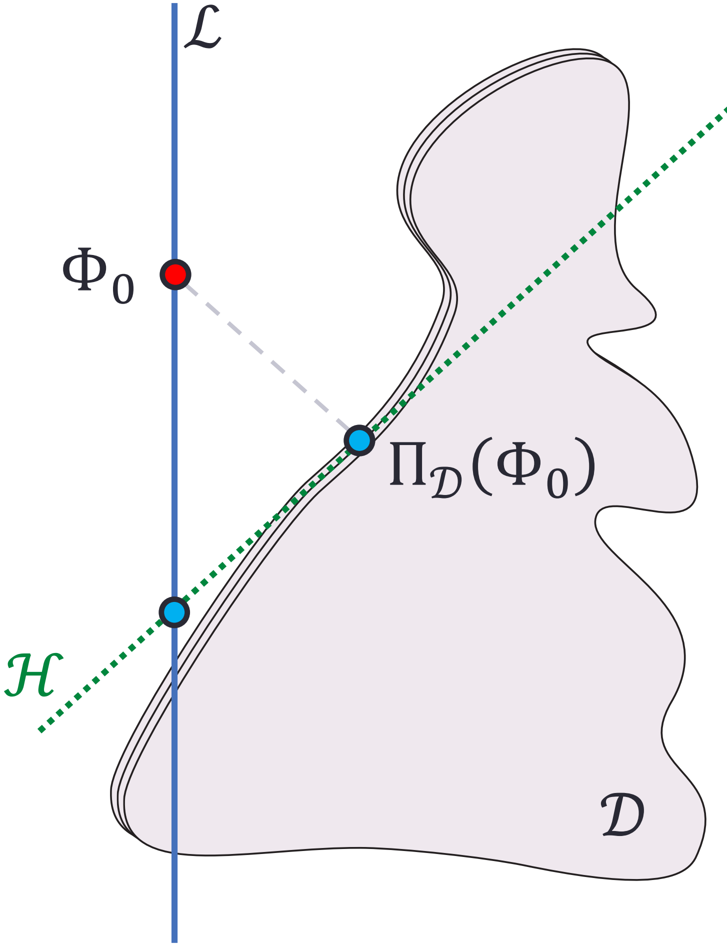


Our Approach

$$\min_{\Phi} \ \|\Phi - \Pi_{\mathcal{D}}(\Phi_0)\|$$
$$\text{s.t.} \quad \Phi \in \mathcal{L}$$
$$\Phi \in \mathcal{H}$$

Solve a linear system

- 2nd-order-**like** convergence
- Parameterless
- Super efficient iterations



Algorithm

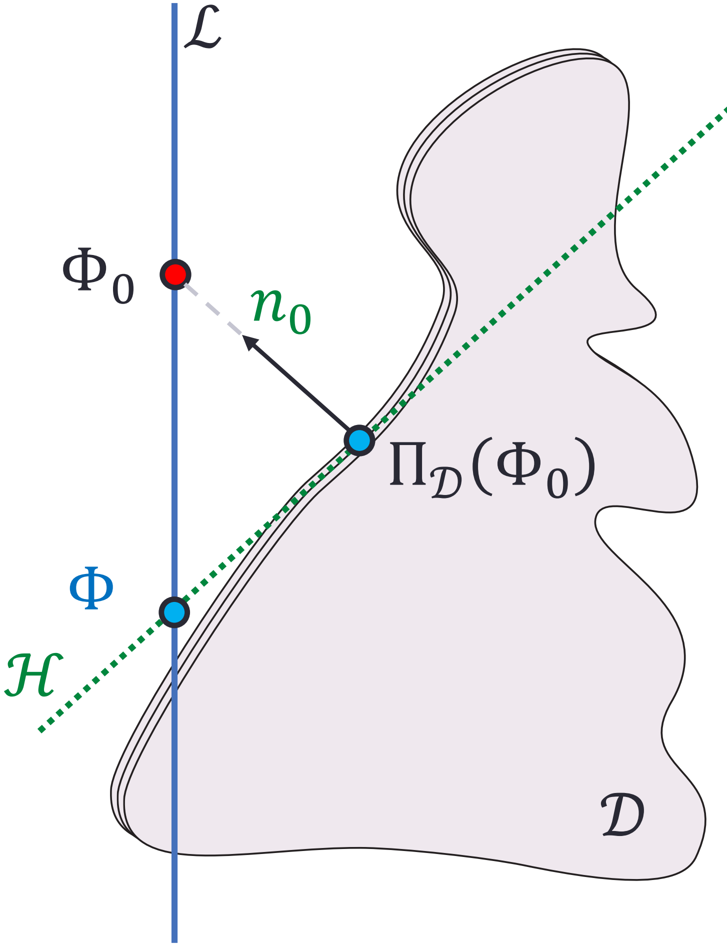
1. Compute $\Pi_{\mathcal{D}}(\Phi_0)$

2. Form the hyperplane:

   $$\mathcal{H} = \{x : n_0 x = n_0 \Pi_{\mathcal{D}}(\Phi_0)\}$$

3. Solve:

   $$\min_{\Phi} \; \|\Phi - \Pi_{\mathcal{D}}(\Phi_0)\|$$
   $$\text{s.t.} \quad \Phi \in \mathcal{L}$$
   $$\Phi \in \mathcal{H}$$

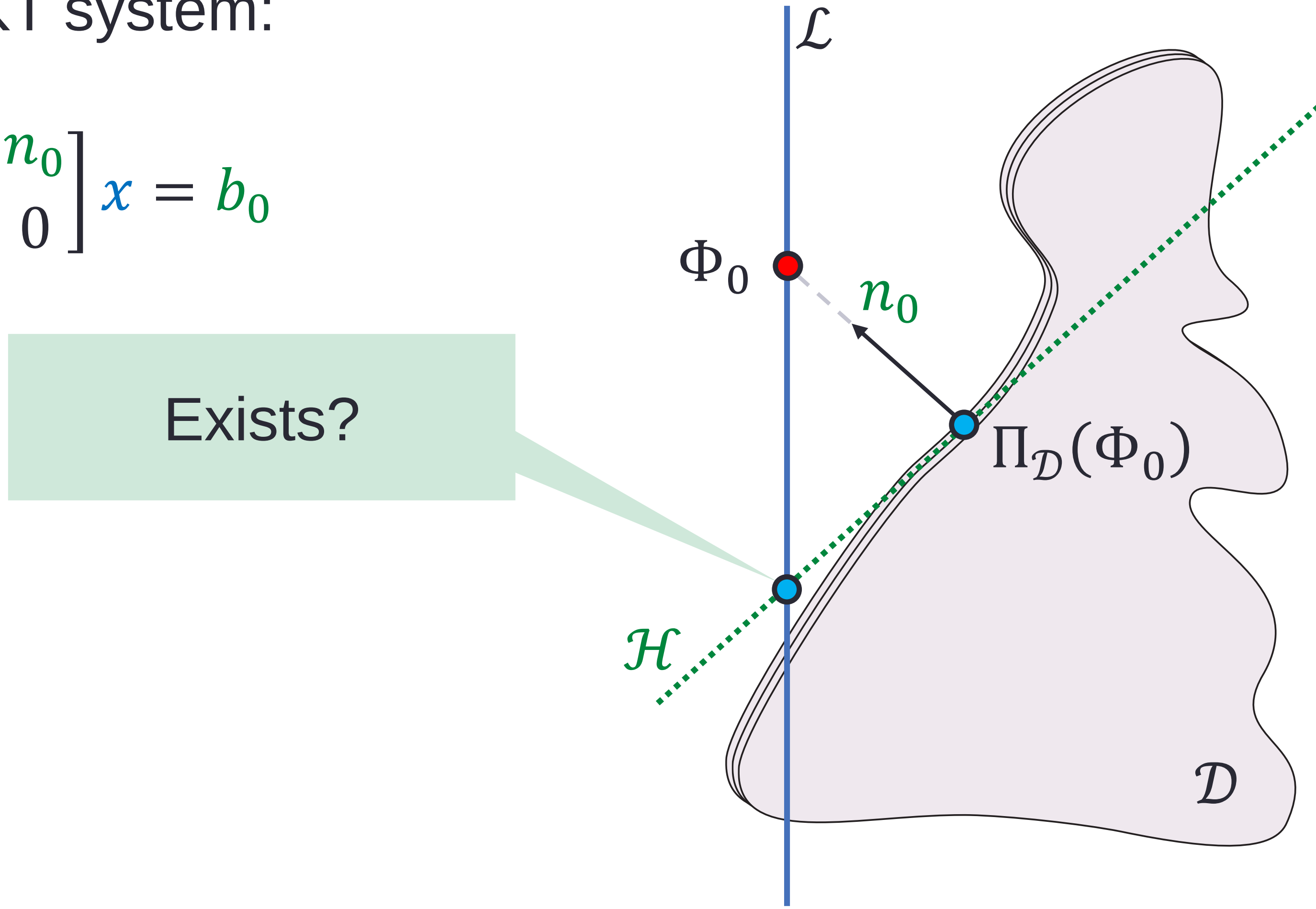


Algorithm

- Equivalent KKT system:

$$\begin{bmatrix} \boldsymbol{M} & n_0 \\ n_0^T & 0 \end{bmatrix} x = b_0$$

- Solvable?



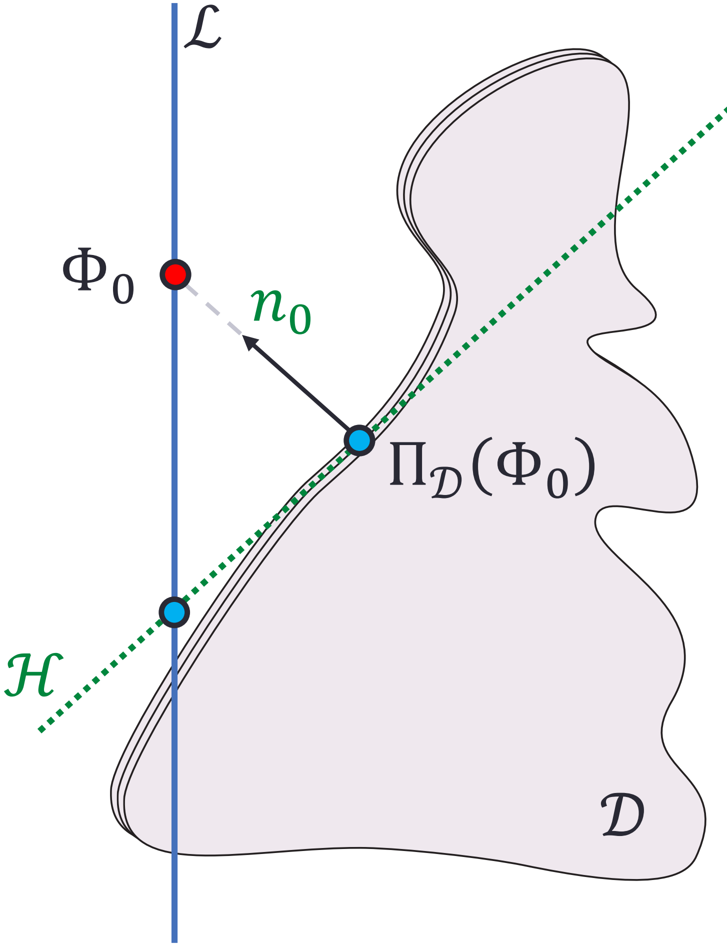
Algorithm

- Equivalent KKT system:

$$\begin{bmatrix} \boldsymbol{M} & n_0 \\ n_0^T & 0 \end{bmatrix} x = b_0$$

- **Theorem**:

  Unique solution exists



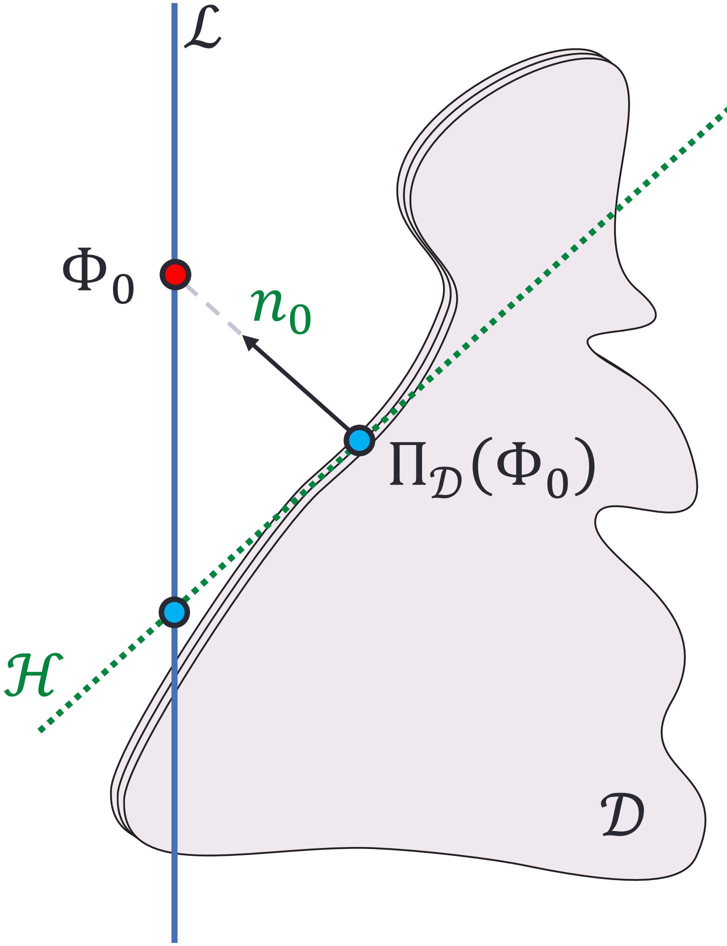
Algorithm

- Equivalent KKT system:

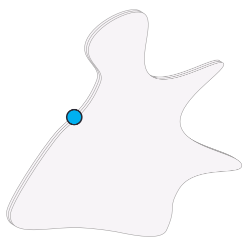
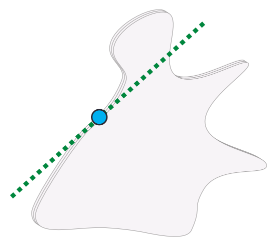
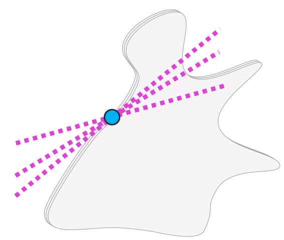
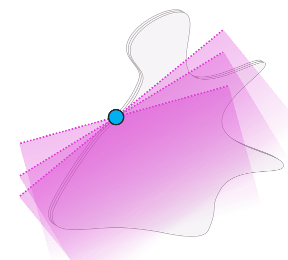
$$\begin{bmatrix} \boldsymbol{M} & n_0 \\ n_0^T & 0 \end{bmatrix} x = b_0$$

FIXED!

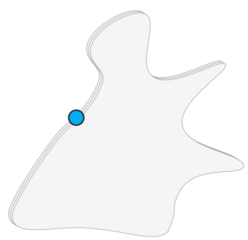
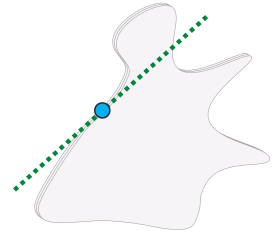
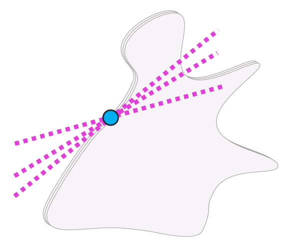
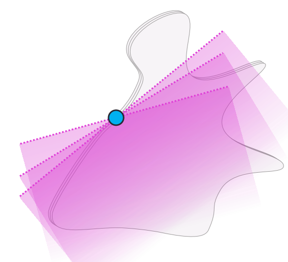
- Efficient pre-factorization
- Closed form expression
- Complexity: **2 back-substitutions**



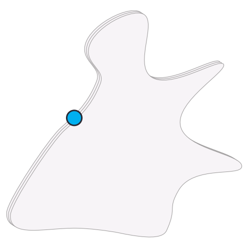
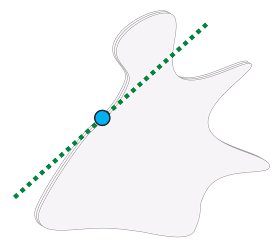
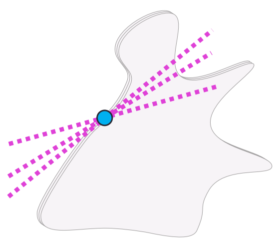
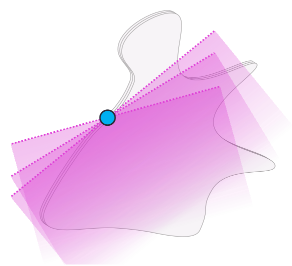
Summary

	Alternating	Ours	Gauss-Newton	Aigerman 2013
Proxy				

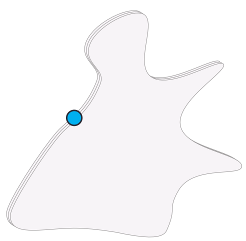
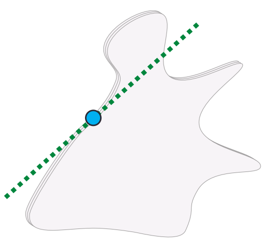
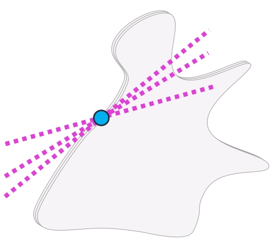
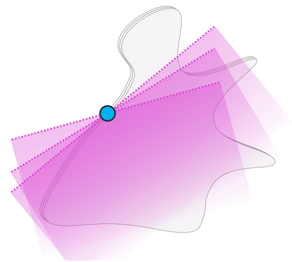
Summary

	Alternating	Ours	Gauss-Newton	Aigerman 2013
Proxy				
Solve	Linear	Linear	Linear	QP

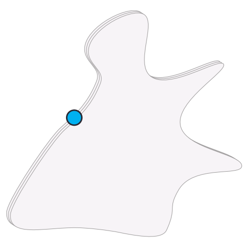
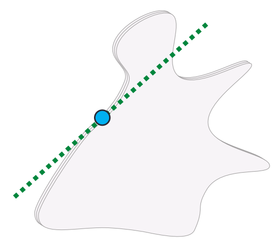
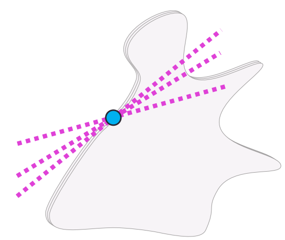
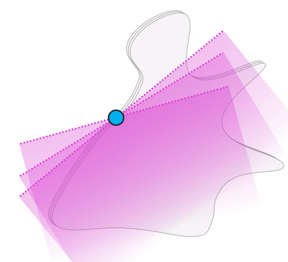
Summary

	Alternating	Ours	Gauss-Newton	Aigerman 2013
Proxy				
Solve	Linear	Linear	Linear	QP
Factorization	Yes	Yes	No	-

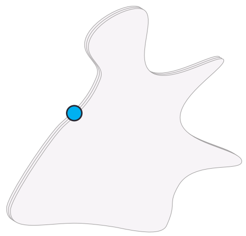
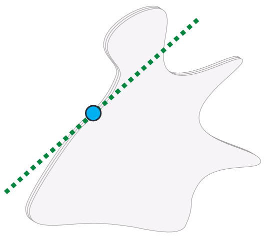
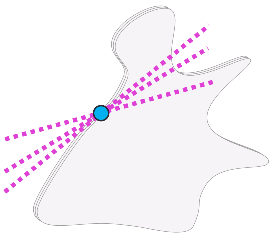
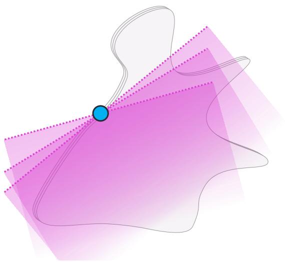
Summary

	Alternating	Ours	Gauss-Newton	Aigerman 2013
Proxy				
Solve	Linear	Linear	Linear	QP
Factorization	Yes	Yes	No	-
Convergence	1^{st}	$\sim 2^{nd}$	$\sim 2^{nd}$	2^{nd}

Summary

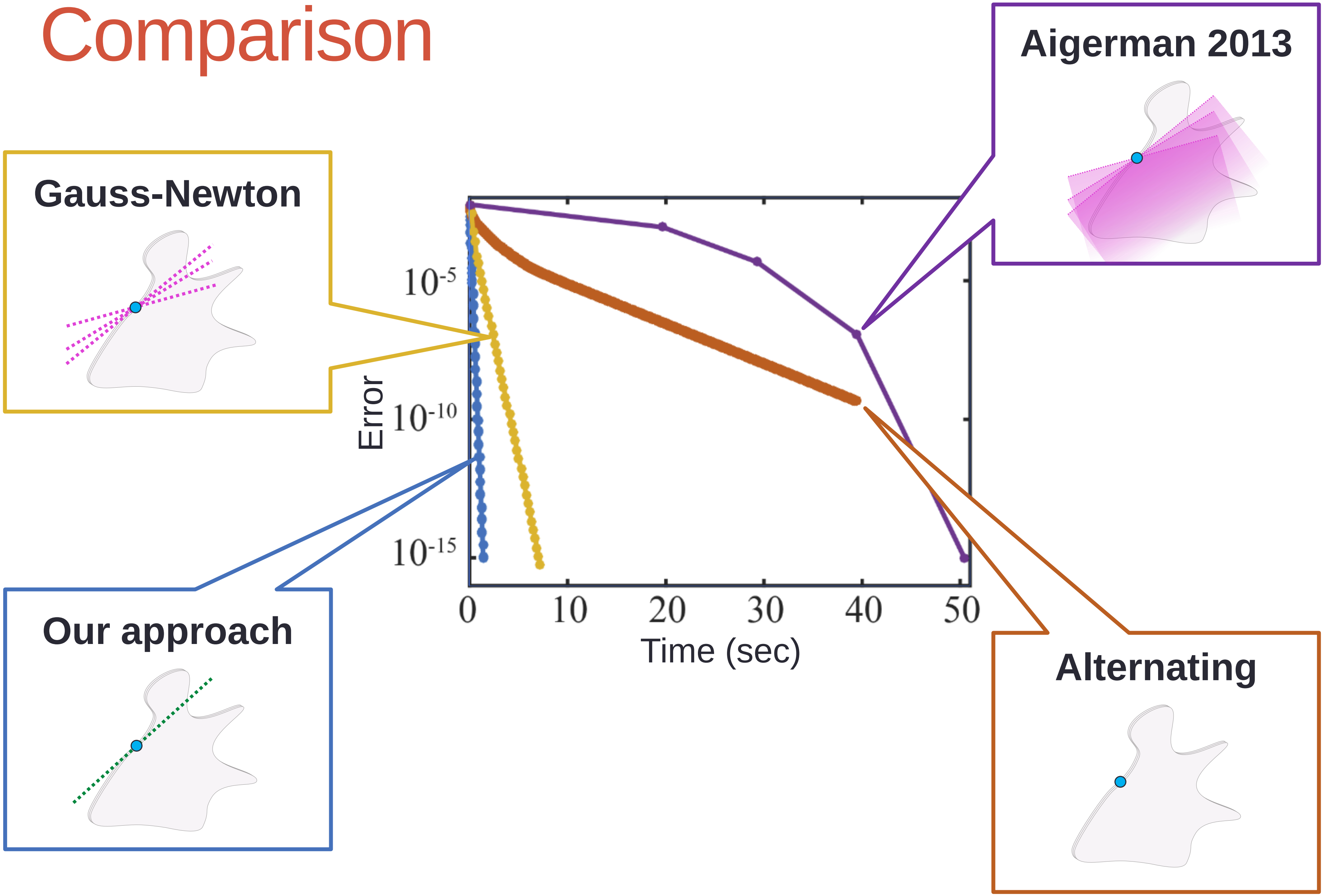
	Alternating	Ours	Gauss-Newton	Aigerman 2013
Proxy				
Solve	Linear	Linear	Linear	QP
Factorization	Yes	Yes	No	-
Convergence	1^{st}	$\sim 2^{nd}$	$\sim 2^{nd}$	2^{nd}
Parameters	No	No	Yes	No

Summary

	Alternating	Ours	Gauss-Newton	Aigerman 2013
Proxy				
Solve	Linear	Linear	Linear	QP
Factorization	Yes	Yes	No	-
Convergence	1^{st}	$\sim 2^{nd}$	$\sim 2^{nd}$	2^{nd}
Parameters	No	No	Yes	No
Guarantee	No	No	No	No

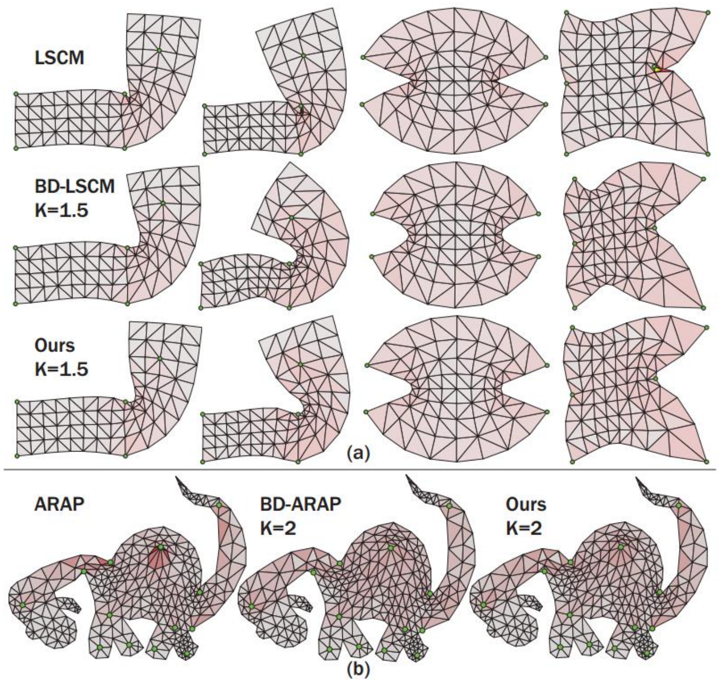
Evaluation and Applications

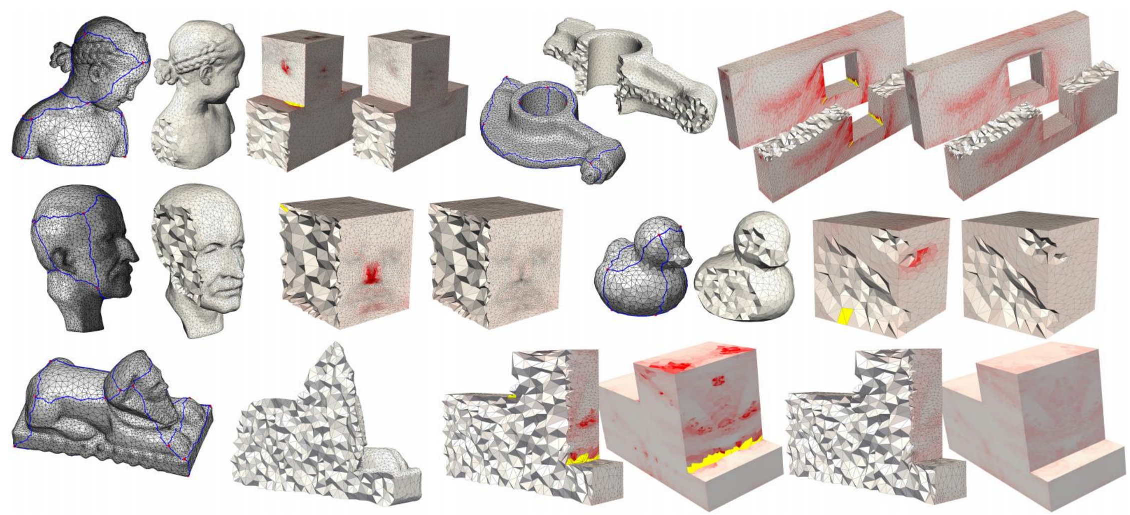
Comparison



Aigerman 2013
Gauss-Newton
Our approach
Alternating
Error
Time (sec)
10-5
10-10
10-15
0
10
20
30
40
50

Comparison with [Aigerman 2013]





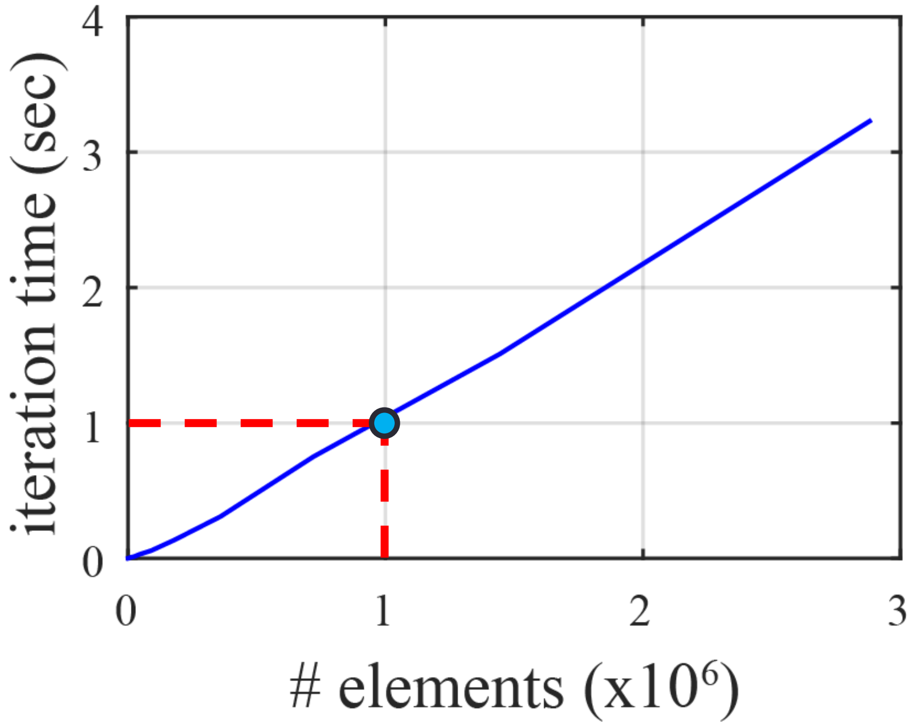
<60,000 elements

Comparable results

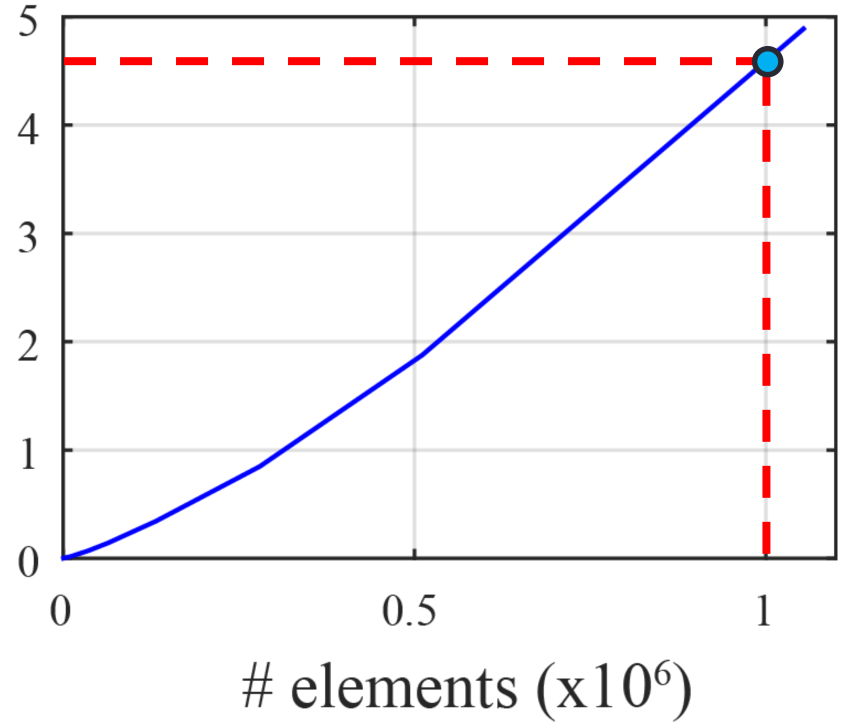
x100 times faster

Scalability

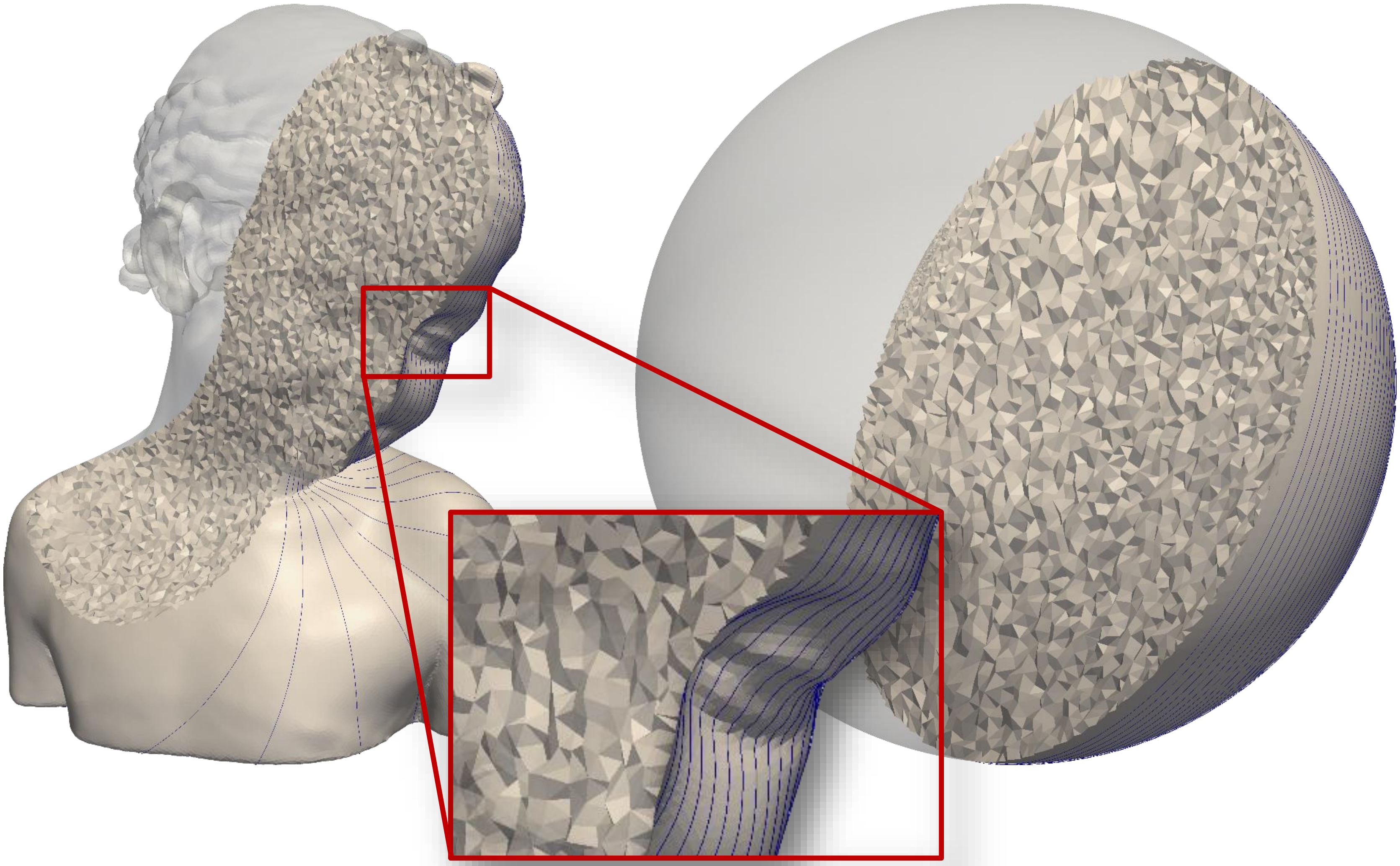
2D



3D

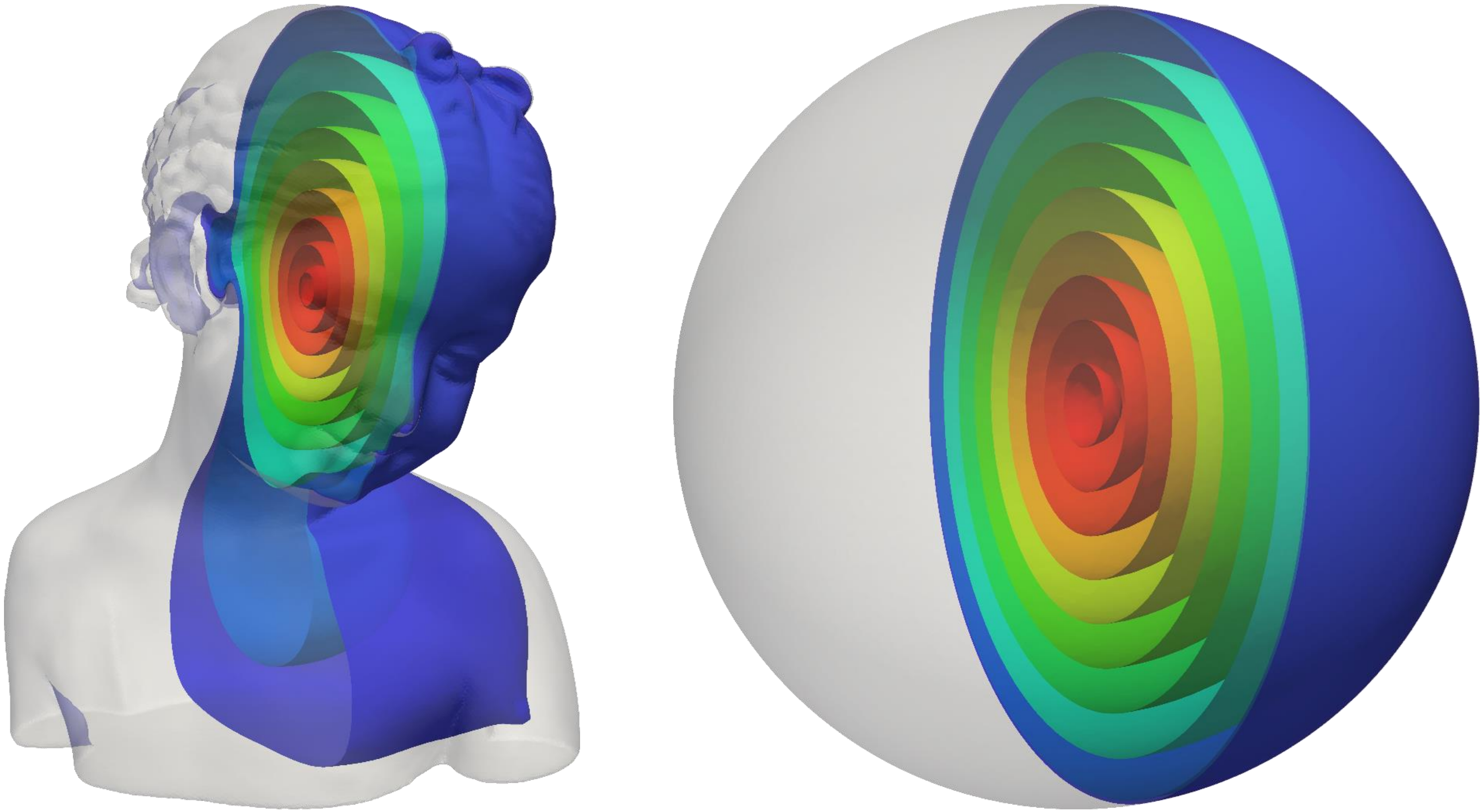


Volumetric Mapping



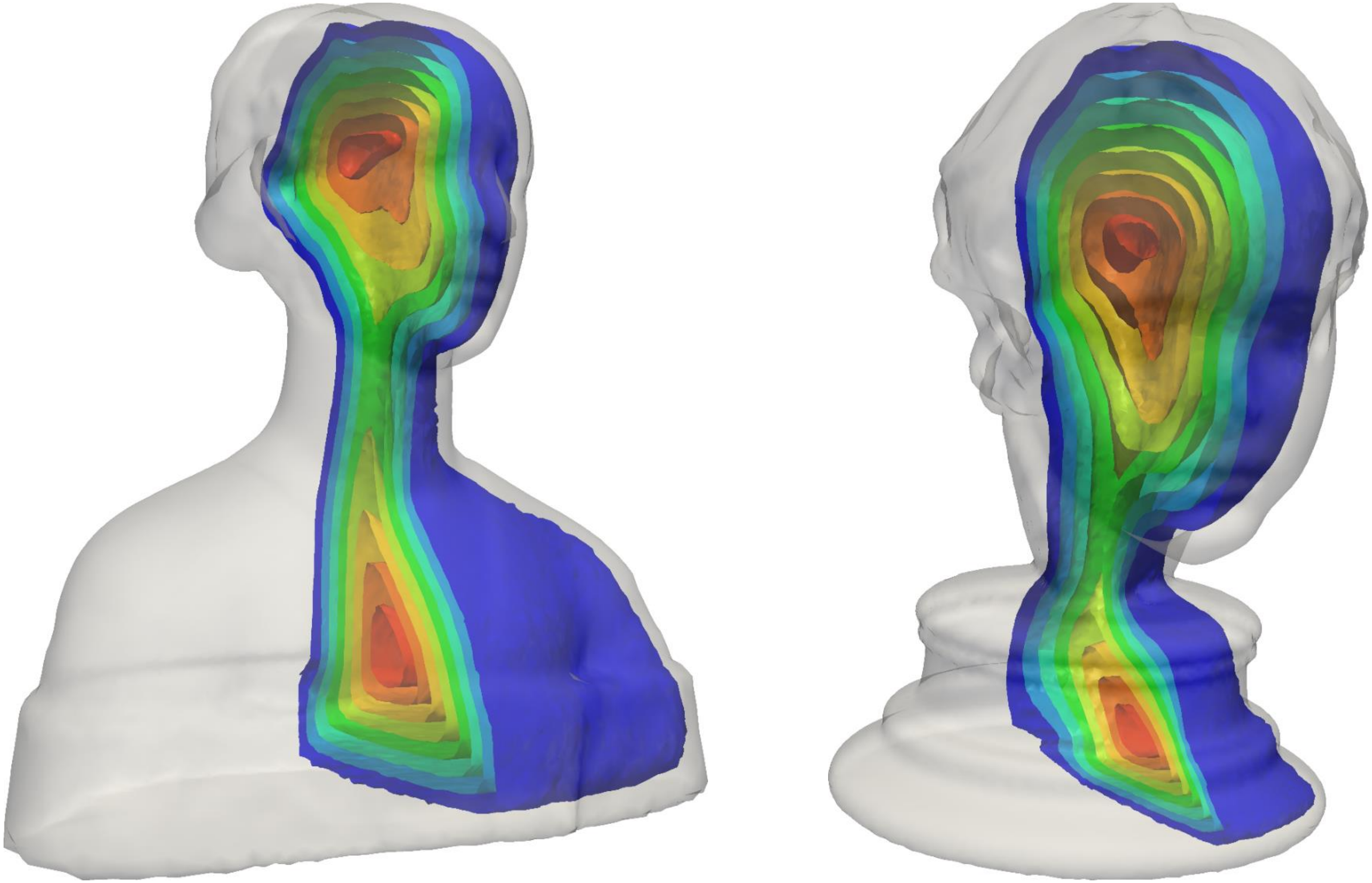
>1,200,000 Tetrahedra

Volumetric Mapping



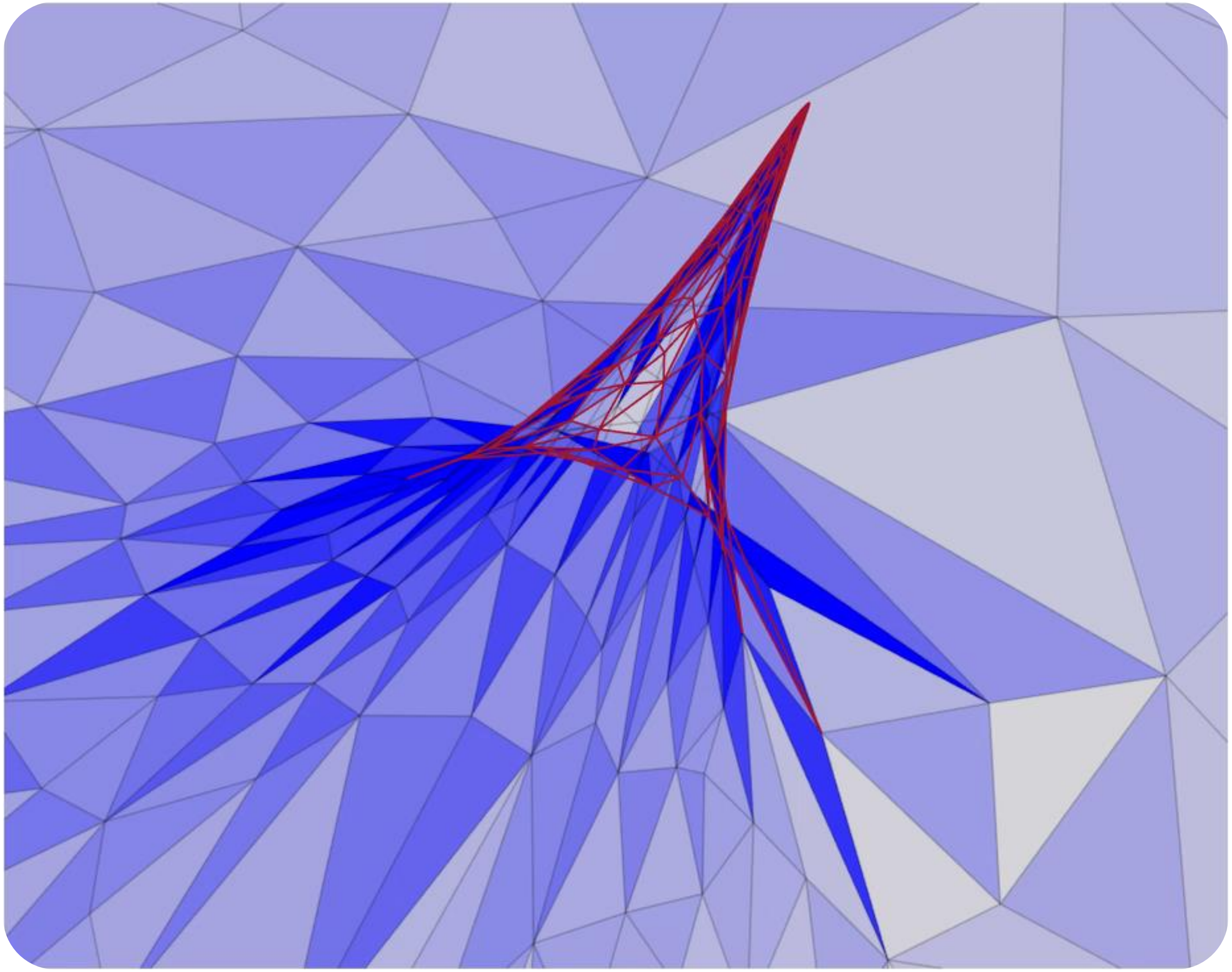
Volumetric Mapping

Volumetric Mapping



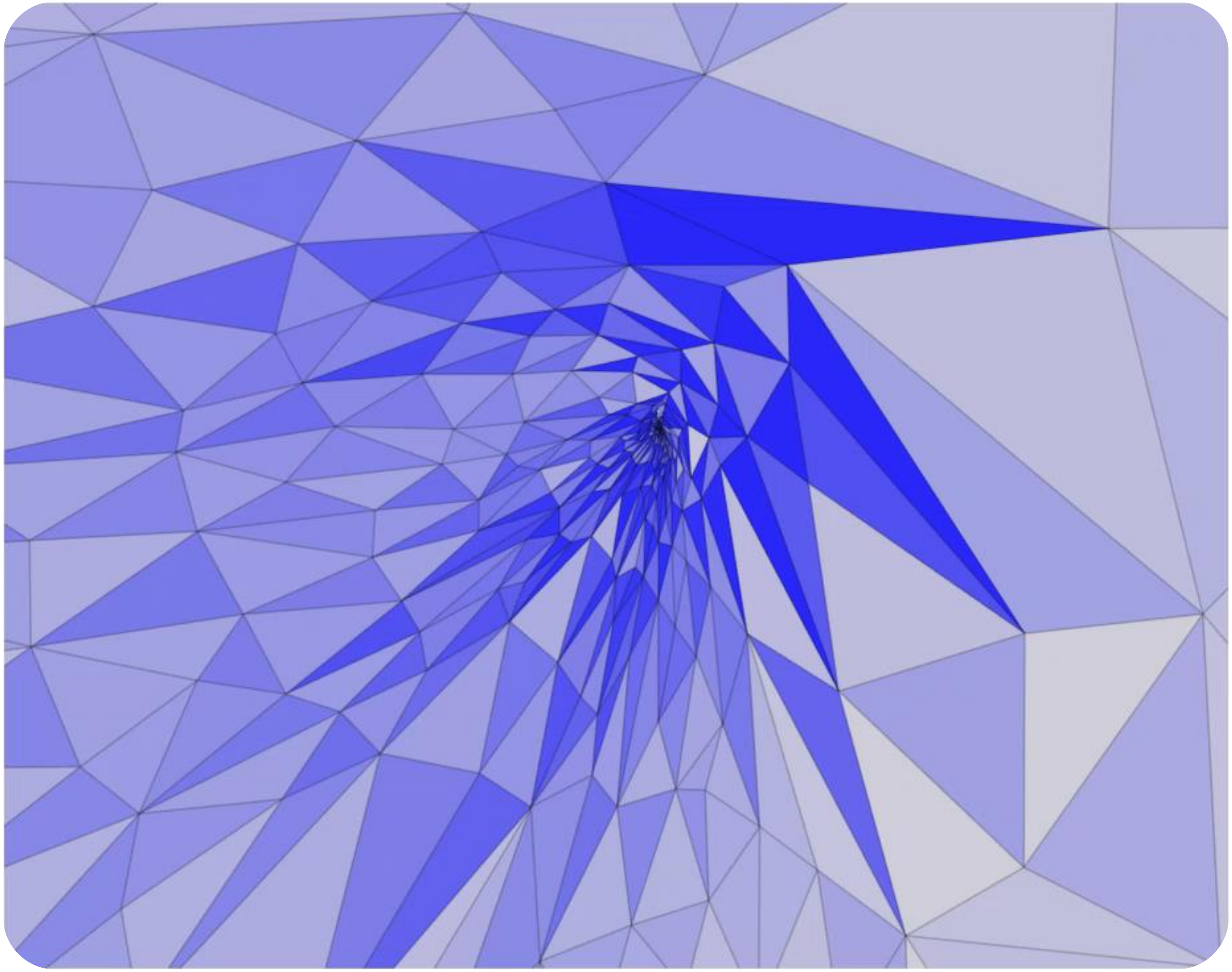
Parameterization

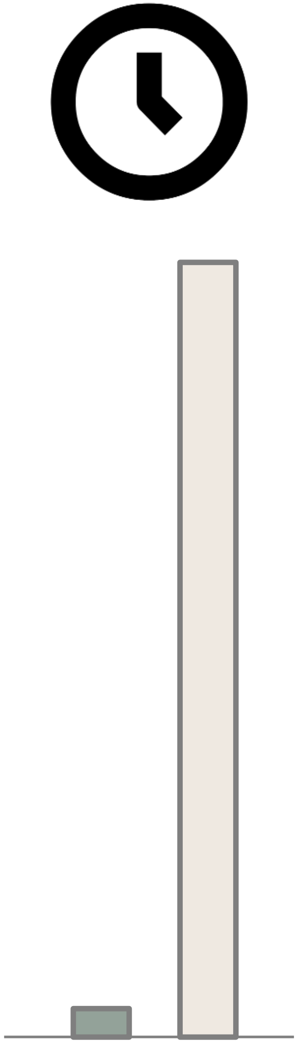
>500,000 Triangles



Flipped Elements

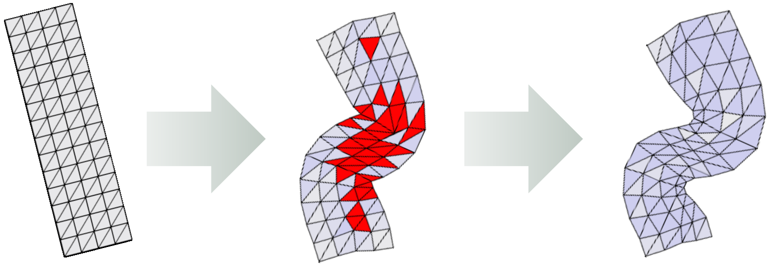
High Distortion



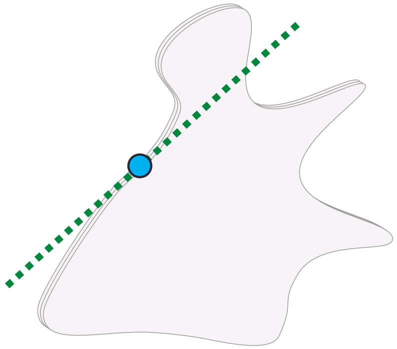


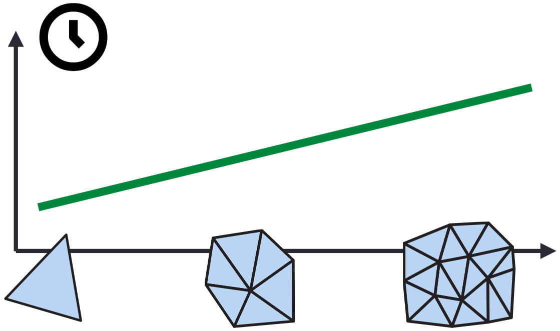
Summary

- Find similar BD mapping



- Simple algorithm
- Efficient and Scalable





- No guarantee
- Optimize other energies?

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Thank you!